GAUGE DEPENDENCE OF RENORMALIZATION CONSTANTS

by

T. P. Beven, B.Sc. (Hons),

A thesis submitted in fulfilment of
the requirements for the degree of

Doctor of Philosophy

in the

University of Tasmania

Hobart

February, 1981.

DECLARATION

Except as stated herein this thesis contains no material which has been accepted for the award of any other degree or diploma in any University. To the best of my knowledge and belief, this thesis contains no copy or paraphrase of material previously published or written by another person, except where due reference is made in the text of the thesis.

T.P. Beven

(T.P. Beven)

ACKNOWLEDGEMENTS

I wish to thank my supervisor, Prof. R.Delbourgo, for his continual and enthusiastic assistance, advice and encouragement throughout the course of this work.

The other members of the Theory Group - Dr. J. Fox, Dr. P. Jarvis, Dr. B. Keck, M. Murray, C. Parker and G. Thompson - have all been very helpful, especially as a most critical audience for the lectures which formed the basis of Chapter 2. My thanks go to them all, and to Dr. Keck in particular for many useful and interesting discussions.

I am grateful to Cheryl Cooper for her cheerful and efficient typing of this thesis.

Finally, thanks to my family, my friends at St. John Fisher College, and other friends for their moral support in the final stages.

This research was financially supported by a University of Tasmania Postgraduate Research Award.

TO MY PARENTS

ABSTRACT

This thesis presents a complete, self-contained list of the renormalization constants Z in unified gauge models to second order in the gauge coupling constant g for four gauge choices. These are the Lorentz covariant (parameter κ) gauges, the axial and Coulomb gauges, and a general non-covariant gauge (parameter a^2/b^2).

Specific details of the calculation of the one-loop infinities are provided. It is thus possible to explicitly verify the Ward-Slavnov-Taylor identities to this order. Comparisons are made which suggest parallelisms between the non-covariant gauges and certain covariant gauges. For instance, the infinities in the axial gauge are exactly what one would find in the Lorentz gauge with $\kappa = -3$. Similarly, less clear correspondences are found between the Coulomb and Fermi ($\kappa = 1$) gauges, and between a general non-covariant gauge with $a^2/b^2 = -3$ and the Landau ($\kappa = 0$) gauge.

Finally, we are able to present for every Lorentz gauge an 'equivalent' non-covariant gauge, and vice-versa.

CONTENTS

FIGURES AND TABLES

I. INTRODUCTION

This chapter is an account of the purpose of the thesis, a placement of its results in the context of related work, and an outline of its general structure. In their own ways, the next two chapters are themselves also introductions - Chapter 2 to the general idea of canonical gauge-fixing, and Chapter 3 to the theory upon which the calculations are based. Thus, a fairly well-rounded initiation into the subject may be gained by perusing all three chapters; whilst a synopsis of the material covered will be found in Sections 1.1 and 1.2, which now follow.

1.1 PREFACE TO THE THESIS

Non-Abelian gauge theory [1,2], expressed in terms of the Yang-Mills [3] field, is an extension of quantum electrodynamics (QED) [4,5,6] in which the gauge field transforms under a semisimple Lie group $SU(n)$ rather than the abelian group $U(1)$. It is a candidate for describing the strong nuclear force and its interactions, through a theory called quantum chromodynamics (QCD) [7]. QCD is mid-way in complexity between QED and Quantum Gravity (QG) [8], and is therefore worth studying not only for its direct physical interpretation, but also for the light it might throw on the latter.

As with all quantum field theories, the basic parameters (masses, coupling constants...) must be redefined through the process of renormalization [1,2]. This comes about owing to the occurrence of infinite matrix elements and associated Lagrangian counter-terms [9]. After renormalization, and barring infrared problems, observable quantities can then be calculated. In this thesis, dimensional regularization (DR)

[10] is used to specify the infinities. This procedure respects the inherent gauge invariance of the theory as expressed in the Ward-Slavnov-Taylor [11,12,13] and Becchi-Rouet-Stora [14] identities.

In conformity with expectations the renormalization constants for QCD, like QED, depend upon the choice of gauge. The thesis is concerned with exploring this gauge dependence by explicit calculation of the renormalization constants Z to first order in perturbation theory of the non-abelian field (second order in the gauge coupling constant g). We choose to do this at first in three distinct gauges - manifestly Lorentz covariant gauges, characterized by an arbitrary parameter κ, and the non-covariant axial and Coulomb gauges, which involve an axis n_μ (which is not k_μ). These three gauges are discussed in ref. [15].

An exact parallelism is uncovered between the values of the various Z in the axial gauge and the covariant gauge with $\kappa=-3$. Reported in ref. [16], this phenomenon has also been noticed by others [17,18]. There is also a less complete correspondence between the scalar and spinor renormalizations for the Coulomb gauge and the covariant gauge with $\kappa=1$ (the Fermi gauge). That such equivalences between non-covariant and covariant gauge choices should occur is a noteworthy fact. This is particularly so in the former case, especially when it is considered that quantization in the covariant gauges requires the introduction of a spurious Faddeev-Popov ghost field [19], whilst the axial gauge is ghost-free [20,21].

To further examine these coincidences the calculations are re-done in a general Frenkel-Taylor gauge [22], characterized by arbitrary parameters a^2 and b^2. This gauge reduces to the covariant gauges when $a^2=b^2=\kappa^{-1/4}$, while a general non-covariant limit is characterized by the ratio a^2/b^2. This in turn reduces to the axial gauge when $a^2/b^2 = 0$ (i.e. $a^2 = 0$, $b^2 \to \infty$), or to the Coulomb gauge when $a^2/b^2 \to \infty$ (i.e. $a^2 \to \infty$, $b^2 = 0$). The results of this investigation were reported in ref. [23].

The infinite parts of the self-energies and vertex corrections are extracted using the Feynman rules for the general gauge. When the three specific gauge limits are taken, we recover the same results as would have been obtained by working from the start in each gauge. This is a useful check. Moreover, however, it is now possible to explore correspondences between covariant gauges and the general non-covariant gauges. We find that scalar and spinor renormalizations are equivalent when $c = (a^2+3b^2)/(a^2-b^2)$. For instance, there is a hitherto unsuspected correspondence between the covariant gauge with $c=0$ (the Laudau gauge) and the non-covariant gauge with $a^2/b^2 = -3$. And one can enumerate an infinity of other such coincidences.

Of all these, though, the axial gauge parallelism still remains the most striking. For it is only in this particular case that <u>all</u> renormalizations are the same (disregarding ghost sources).

1.2 THE THESIS IN CONTEXT

In 1954 Yang and Mills [3] introduced a generalization of the electromagnetic field, the 'Yang-Mills' field, which undergoes non-abelian $SU(2)$ gauge transformations. Canonical quantization of the non-abelian theory in the radiation (Coulomb) and spatial (axial) gauges was undertaken by Arnowitt and Fickler [24], and Schwinger [25,26], who also concerned himself with the Lorentz gauges. The Feynman rules [27] for the theory, incorrect until the introduction of the ghost field, are now well-understood [15,19].

The study of the subject accelerated when possible applications in weak and strong interactions were realized [28, 29]. For good reviews of the problems encountered in the regularization and renormalization of Yang-Mills theory the reader should consult refs. [1] and [2].

Now, it is important that in such theories characterized by so strong a reliance on gauge invariance that a regularization scheme should be adopted which respects this property. Such a program is provided by dimensional regularization, first put forward in 1972 by Ashmore [30], Cicuta and Monfaldi [31], and 't Hooft and Veltman [32]. A complete review of dimensional regularization and a comparison with other schemes can be found in ref. [10].

As far as observable quantities are concerned, their calculation may proceed in a variety of gauges : either ones in which the Lorentz covariance is maintained throughout, or others which sacrifice this manifest covariance in return for more direct physical interpretations. In the covariant gauges considered here and the Coulomb gauge, it is necessary to introduce a fictitious ghost field into the path-integral quantization procedure, as was first pointed out by Faddeev and Popov [19], following earlier suggestions by Feynman [33].

The axial gauge, on the other hand, is well known to be ghost-free [20,21], and is useful because the Ward identities then take on their naive form [18,34]. Outstanding problems of unitarity and the like have been overcome in the axial gauge, and it now rests on a firm footing [35,36,37,38,39, 40,41,42]. It is, however, not the only gauge which precludes the existence of Faddeev-Popov ghosts. Konetschny [43] has investigated the ghost-free gauges, and discovered a general class of axial-type gauges which may exhaust all possibilities. Recently, Harrington and Tabb [44] have expanded the known group of ghost-free gauges by considering gauge restraints on the non-abelian electric field.

In 1973, Gross and Wilczek [45] and Politzer [46] made the remarkable observation that non-abelian gauge theories are asymptotically free. That is, their ultraviolet behaviour approaches that of free field theory. Such theories will exhibit canonical scaling behaviour [7,47,48,49,50] in the ultraviolet limit (apart from logarithmic corrections), and this makes them candidates for describing the strong interactions through quantum chromodynamics.

QCD is a gauge theory of interacting constituents or *quarks*. The quarks are bound together in hadrons by the exchange of *gluons*. In the large momentum limit, the property of asymptotic freedom allows the older parton model to be used, whilst perturbative QCD provides radiative corrections to this simple picture. The quarks are usually presumed to have spin-1/2, but experimental results so far do not totally exclude the possibility of some small fraction possessing spin-0 [48].

Much of the contemporary research in QCD is centered around the subject of quark confinement. It is thought that the infrared singularities caused by the self-interaction of gluons will lead to complete bondage of the quarks [48,50,51]. The planar gauge (which is closely related to the axial gauge) is particularly useful for performing QCD structure functions calculations since it allows for a parton interpretation of Feynman diagrams [50].

The cause of asymptotic freedom has been investigated in the Lorentz covariant [52], Coulomb [22, 53] and axial [22] gauges. In the course of these inquiries, some renormalization constants have been evaluated. The present work completes this list and compares the values of these various constants in different gauges.

This thesis grew out of an interest in the covariance of pole parts in the Coulomb and axial gauges. Hagen and Singh [54] have claimed that the Coulomb gauge yields non-covariant scalar and spinor wave-function renormalizations to fourth order in the coupling constant in a two-dimensional model. Whilst attempting to refute this claim we received a paper by Heckathorn [55] which gives an inductive proof for the covariance of pole parts for the Coulomb gauge in QED. We have tried to extend this proof to the axial gauge, but were hampered by the differing structures of the vector meson propagators. It was a combination of this work, and a study of that by Frenkel and Taylor [22], which led to the realization that it would be possible to perform calculations in a general gauge, and return to the other three distinct gauges in certain limits.

1.3 STRUCTURE OF THE THESIS

In substance this thesis comprises five chapters. Chapter 2 is, in a sense, strongly decoupled from the rest of the thesis. This is because it deals with canonical gauge fixing of the free Maxwell field, and the problems associated therewith. It should be viewed as a means of 'getting one's feet on the ground' as far as the concept of gauge invariance is concerned. Its rather pedagogic nature derives from its origin as a series of graduate lectures.

THESIS FLOW CHART

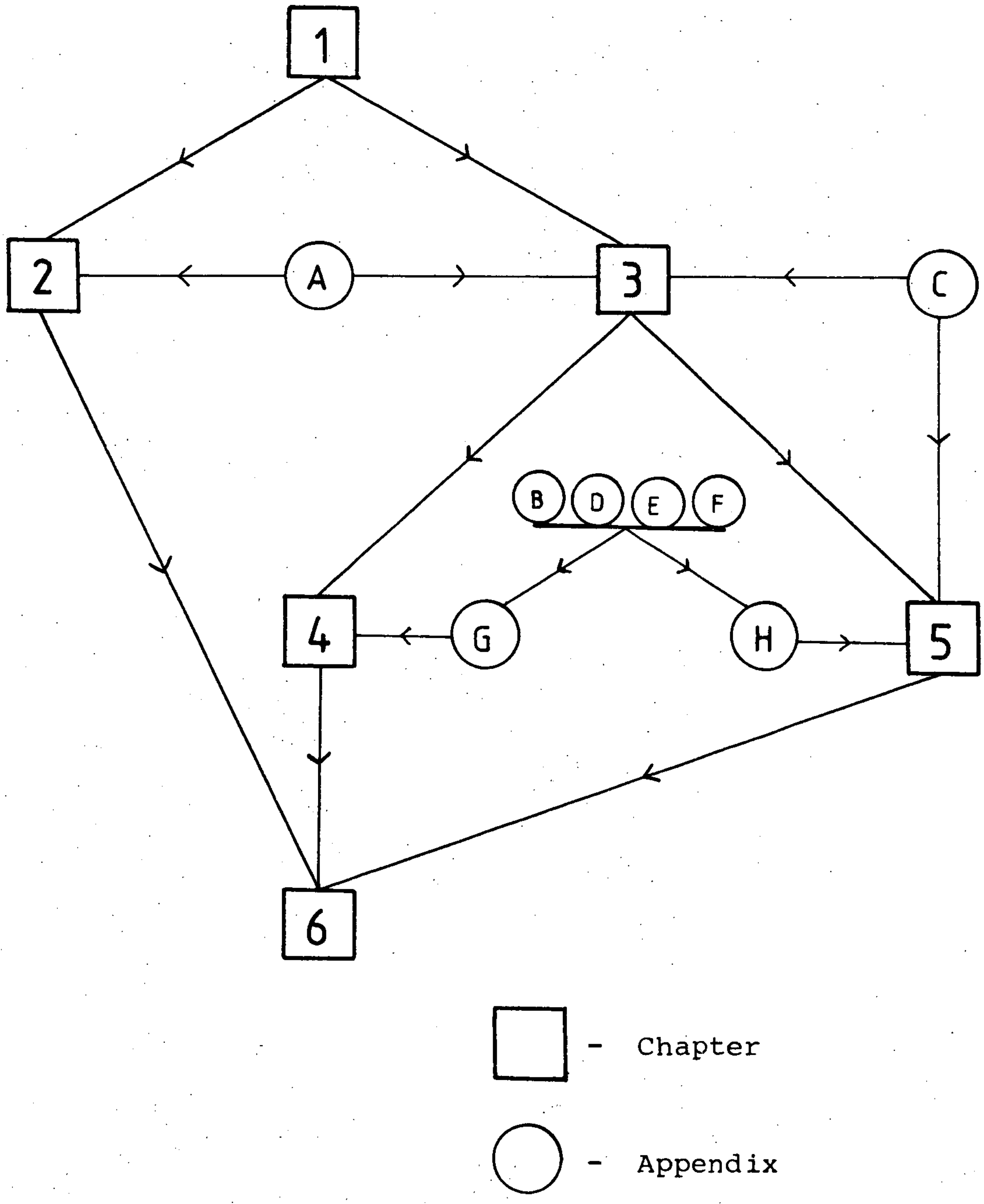


FIGURE 1.1 A flow chart showing the structure of the thesis; in particular, the relationships between chapters and appendices are indicated.

Chapter 3 is concerned with the background theory (path-integral quantization, Feynman rules and the like) upon which the calculations of renormalization constants are based.

It is in Chapters 4 and 5 that the crux of the thesis rests. In the former, renormalization constants are derived in the three specific gauge choices. In the latter, the corresponding results are obtained in the general Frenkel-Taylor gauge. In particular, Tables 4.11, 4.12, 5.1 and 5.2 summarize the main achievements of the work contained herein.

Chapter 6 rounds off the thesis by presenting an overview of the work, and the outlook for future research.

Appendix A specifies the notation used. The other appendices contain details of the calculations.

It has been found easiest to reference the thesis by chapter rather than as a whole. Whilst this leads to some repetition, it also means that each chapter is somewhat self-contained, within the confines of its intended purpose.

As a useful guide, Figure 1.1 gives a 'flow chart' of the thesis so that the relationships between the various chapters and the appendices may be seen at a glance.

* * * * * * * * * *

REFERENCES - CHAPTER 1

1. E.S. Abers and B.W. Lee : Phys. Rep. 9, 1 (1973).
2. G. Costa and M. Tonin : Rev. Nuovo Cim. 5, 29 (1975).
3. C.N. Yang and R.L. Mills : Phys. Rev. 96, 191 (1954).
4. J.D. Bjorken and S.D. Drell : *'Relativistic Quantum Field Theory,'* McGraw-Hill (U.S.A.) (1964).
5. A.I. Akheizer and V.B. Beretstetskii : *'Quantum Electrodynamics,'* Interscience Publishers (N.Y.) (1965).
6. J.M. Jauch and F. Rohrlich : *'The Theory of Photons and Electrons,'* Springer-Verlag (N.Y.) (1976).

7. W. Marciano and H. Pagels : Phys. Rep. 36, 137 (1978).

8. C.J. Isham, R. Penrose and D.W. Sciama (eds.) : *'Quantum Gravity - an Oxford Symposium,'* Clarendon Press (Oxford) (1975).

9. R. Delbourgo : Rep. Prog. Phys. 39, 345 (1976).

10. G. Leibbrandt : Revs. Mod. Phys. 47, 849 (1975).

11. J.C. Ward : Phys. Rev. 78, 1824 (1950).

12. A.A. Slavnov : Theor. Math. Phys. 10, 99 (1972).

13. J.C. Taylor : Nucl. Phys. B33, 436 (1971).

14. C. Becchi, A. Rouet and R. Stora : Commun. Math. Phys. 42, 127 (1975).

15. E.S. Fradkin and I.V. Tyutin : Phys. Rev. D2, 2841 (1970).

16. T.P. Beven and R. Delbourgo : Lett. Nuovo Cim. 23, 433 (1978).

17. T. Curtright and G. Ghandour : Ann. Phys. 112, 237 (1978).

18. R. Delbourgo, A. Salam and J. Strathdee : Nuovo Cim. A23, 237 (1974).

19. L.D. Faddeev and V.N. Popov : Phys. Lett. B25, 29 (1967).

20. W. Kummer : Acta Phys. Austriaca 41, 315 (1974).

21. J. Frenkel : Phys. Rev. D13, 2325 (1976).

22. J. Frenkel and J.C. Taylor : Nucl. Phys. B109, 439 (1976).

23. T.P. Beven and R. Delbourgo : Lett. Nuovo Cim. 27, 565 (1980) and erratum, 28, 400 (1980).

24. R.L. Arnowitt and S.I. Fickler : Phys. Rev. 127, 1821 (1962).

25. J. Schwinger : Phys. Rev. 127, 324 (1962).

26. J. Schwinger : Phys. Rev. 130, 402 (1963).

27. R.P. Feynman : Phys. Rev. 76, 769 (1949).

28. J. Bernstein : Revs. Mod. Phys. 46, 7 (1974).

29. S. Weinberg : Revs. Mod. Phys. 46, 255 (1974).

30. J.F. Ashmore : Lett. Nuovo Cim. 4, 289 (1972).

31. G.M. Cicuta and E. Montaldi : Lett. Nuovo Cim. 4, 329 (1972).

32. G. 't Hooft and M. Veltman : Nucl. Phys. B44, 189 (1972).

33. R.P. Feynman : Acta Phys. Polonica 24, 697 (1963).

34. J. Bernstein : Phys. Rev. D15, 2273 (1977).

35. W. Kainz, W. Kummer and M. Schweda : Nucl. Phys. B79, 484 (1974).

36. W. Konetschny and W. Kummer : Nucl. Phys. B100, 106 (1975).

37. W. Konetschny and W. Kummer : Nucl. Phys. B108, 397 (1976).

38. J. Frenkel : Nuovo Cim. A32, 313 (1976).

39. J. Frenkel and R. Meuldermans : Phys. Lett. B65, 64 (1976).

40. W. Konetschny and W. Kummer : Nucl. Phys. B124, 145 (1977).

41. W. Konetschny : Nuovo Cim. A44, 465 (1978).

42. W. Konetschny : Nucl. Phys. B152, 153 (1979).

43. W. Konetschny : Phys. Lett. B90, 263 (1980).

44. B.J. Harrington and C.H. Tabb : Univ. of California preprint NSF-ITP-80-25 (1980).

45. D.J. Gross and F. Wilczek : Phys. Rev. Lett. 30, 1343 (1973).

46. H.D. Politzer : Phys. Rev. Lett. 30, 1346 (1973).

47. C.H. Llewellyn-Smith : Oxford preprint 54/79 (1979).

48. E.L. Berger : Argonne preprint ANL-HEP-CP-79-51 (1979).

49. C. Quigg : Fermi N.A.L. preprint FERMILAB-Conf.-80/64-THY, (1980).

50. Yu. L. Dokshitzer, D.I. Dyakonov and S.I. Troyan : Phys. Rep. 58, 269 (1980).

51. F.J. Yundurain : Madrid Univ. preprint FTUAM/80-7, (1980).

52. D.J. Gross and F. Wilczek : Phys. Rev. D8, 3633 (1973).

53. A. Ali, J. Bernstein and A. Zepeda : Phys. Rev. D12, 503 (1975).

54. C.R. Hagen and L.P.S. Singh : Phys. Rev. D21, 1620 (1980).

55. D. Heckathorn : Nucl. Phys. B156, 328 (1979).

2. GAUGE CHOICES IN QUANTUM ELECTRODYNAMICS

This chapter has a two-fold purpose: firstly, to introduce the non-specialist reader to the concept of choosing a gauge in classical electromagnetism and the techniques used to carry this idea over into quantum field theory; and secondly, to provide a short, by-no-means complete, review of some texts and papers bearing on the topic. The free Maxwell field will be quantized via the so-called canonical procedure in a variety of gauges, each of which has its own special difficulties and benefits.

2.1 THE CLASSICAL FREE MAXWELL FIELD

It is well-known that the components of the electromagnetic field strengths $\underline{E}$ and $\underline{B}$ form an antisymmetric field tensor with components [1,2,3]:

$$F^{\mu\nu} = \begin{bmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & B^3 & -B^2 \\ -E^2 & -B^3 & 0 & B^1 \\ -E^3 & B^2 & -B^1 & 0 \end{bmatrix} \qquad \ldots(2.1)$$

i.e.

$$\left.\begin{aligned} F^{0j} &= E^j \\ F^{ij} &= \epsilon^{ijk} B^k \end{aligned}\right\} \qquad \ldots(2.2)$$

where ϵ^{ijk} is the unit antisymmetric tensor.

Maxwell's equations can be separated into two distinct pairs, namely

$$\nabla \times \underline{E} = -\dot{\underline{B}} \qquad \ldots(2.3a)$$

$$\nabla . \underline{B} = 0 \qquad \ldots(2.3b)$$

and

$$\nabla . \underline{E} = 0 \quad \text{(Gauss's law)} \qquad \ldots(2.4a)$$

$$\nabla \times \underline{B} = \dot{\underline{E}} \qquad \ldots(2.4b)$$

The first pair, (2.3) can be written as

$$\partial^{\rho} F_{\mu\nu} + \partial^{\mu} F_{\nu\rho} + \partial^{\nu} F_{\rho\mu} = 0 \qquad \text{...(2.5)}$$

and the second pair, (2.4), as

$$\partial^{\nu} F_{\mu\nu} = 0 \qquad \text{...(2.6)}$$

(2.3) leads to a natural expression for the fields $\underline{E}$ and $\underline{B}$ in terms of a four-potential $A_\mu = (A_0, \underline{A})$, so that

$$\underline{E} = -\nabla A_0 - \dot{\underline{A}} \qquad \text{...(2.7a)}$$

and

$$\underline{B} = \nabla \times \underline{A} \qquad \text{...(2.7b)}$$

or, equivalently,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad \text{...(2.8)}$$

It follows from (2.5) and (2.6) that each component of the tensor field satisfies the wave equation

$$\Box F_{\mu\nu} = 0 \qquad \text{...(2.9)}$$

which, in terms of the vector potential, reads [4]

$$\Box A_\nu - \partial_\nu \chi = 0 \qquad \text{...(2.10)}$$

with

$$\chi \equiv \partial^{\mu} A_\mu \qquad \text{...(2.11)}$$

The first pair of Maxwell's equations, (2.5), have been incorporated by definition, as it were, into (2.8). The second pair, (2.6), can be obtained via an action principle using the Lagrangian (density) [1, 2, 3, 5, 6, 7]:

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} \qquad \text{...(2.12)}$$

It is here that we introduce the concept of a change of gauge. Clearly, the vector potential is not uniquely determined by (2.8), for this equation remains unaltered under

$$A_\mu \rightarrow A'_\mu = A_\mu + \partial_\mu \Lambda \qquad \text{...(2.13)}$$

where Λ is an arbitrary function for which the gradient exists.

(2.13) is referred to as a gauge transformation. A particular choice of Λ does not affect any observable quantities, since these depend only on the field $F_{\mu\nu}$ (i.e. $\underline{E}$ and $\underline{B}$). (2.12) is in fact the simplest Lagrangian which is both relativistically and gauge invariant, and which leads to linear field equations [1].

Now, to leap ahead of ourselves slightly, it is a fact that the massless, spin 1 photon has only two states of polarization [5]. This leads to formal difficulties when describing the electromagnetic field in terms of a four component potential A_μ. Quite simply, limitations must be placed on the A-field which remove two of these degrees of freedom. It is precisely the freedom in the choice of gauge function Λ in (2.13) which allows this to be done. In particular, a *gauge condition* may be imposed which either restricts one or more of the components of A_μ to particular values or which relates them to each other in some way, or both. This condition specifies the *gauge* in which one works. When making the transition to a quantum mechanical description of the field one particular gauge condition may appear simpler to use than another, but classically there is little distinction between gauges.

We will describe here in some detail the two preferred classical choices, the so-called Lorentz and Coulomb gauges, so that when quantizing it will not seem that either of these, or any other gauge, is peculiar only to the quantum field. The former is a manifestly relativistically invariant gauge choice, whilst the latter sacrifices this invariance in return for an immediate and natural decomposition of the four potential A_μ into an electrostatic potential A_0 (which is zero in the free field case) and a transverse wave contribution .

LORENTZ GAUGE

The simplest relativistically invariant condition which may be imposed on the A-field is the divergence or *Lorentz* condition.

$$\partial^\mu A_\mu(x) \equiv \chi(x) = 0 \qquad \text{...(2.14)}$$

This restricts the gauge function in (2.13) to a solution of the wave equation

$$\Box \quad \Lambda(x) = 0 \qquad \ldots(2.15)$$

and ensures that the potential field also satisfies

$$\Box \quad A_\mu(x) = 0 \qquad \ldots(2.16)$$

The freedom in the choice of gauge function under (2.15) may now be used to eliminate two of the components of A_μ , leaving only two dynamical variables [5] .

(2.16) is satisfied by the plane wave form:

$$A_\mu(x) = \int \frac{d^3k}{2k_0} \, \epsilon_{\mu\lambda}(k) \left[a_\lambda(k) \, e^{-ik.x} + a_\lambda^+(k) \, e^{ik.x} \right] \qquad \ldots(2.17)$$

$$k^2 = 0 \,, \quad k_0 = |\underline{k}|$$

where the $\epsilon_{\mu\lambda}(k)$ are polarization vectors distinguished by the index λ , and the $a_\lambda(k)$ are amplitudes. It should be noted that the sum over λ is an ordinary one. The Lorentz condition (2.14) implies that

$$k^\mu \, a_\mu(k) = 0 \qquad \ldots(2.18)$$

where

$$a_\mu(k) \equiv \epsilon_{\mu\lambda}(k) \, a_\lambda(k) \qquad \ldots(2.19)$$

By a judicious selection of the components of the polarization matrix it is now possible to leave $a_1(k)$ and $a_2(k)$ unrestricted, whilst $a_0(k)$ and $a_3(k)$ can be made to depend on each other. Such a possibility is realized if:

$$\left.\begin{array}{ll} k^i \, \epsilon_{i\lambda}(k) = 0 & \lambda = 1, 2 \\ \epsilon_{0\lambda}(k) = 0 & \\ & \\ \text{and} \quad \epsilon_{i3}(k) = \dfrac{k_i}{k_0} & \\ \epsilon_{03}(k) = 0 & \\ \epsilon_{i0}(k) = 0 & \\ \epsilon_{00}(k) = 1 & \end{array}\right\} \qquad \ldots(2.20)$$

Thus

$$\left.\begin{array}{l} k^\mu \, \epsilon_{\mu 3}(k) = -k_0 \\ k^\mu \, \epsilon_{\mu 0}(k) = k_0 \end{array}\right\} \qquad \ldots(2.21)$$

Whence, using (2.18),

$$a_3(k) - a_0(k) = 0 \qquad \ldots(2.22)$$

Now that this connection between the 'scalar' and 'longitudinal' amplitudes has been established, a gauge function satisfying (2.15) is used to remove these components from the theory. That is,

$$\Lambda(x) = \int \frac{d^3k}{2k_0} \left[\tilde{\Lambda}(k) e^{-ik.x} + \tilde{\Lambda}^*(k) e^{ik.x} \right] \qquad \ldots(2.23)$$

leads via (2.13) to the transformed potential

$$A'_\mu(x) = \int \frac{d^3k}{2k_0} \left[(a_\mu(k) - i k_\mu \tilde{\Lambda}(k)) e^{-ik.x} + (a_\mu^*(k) + i k_\mu \tilde{\Lambda}^*(k)) e^{ik.x} \right] \qquad \ldots(2.24)$$

The old and new amplitudes are related by

$$a'_\mu(k) - a_\mu(k) = \epsilon_{\mu\lambda}(k) \left[a'_\lambda(k) - a_\lambda(k) \right] = -i k_\mu \tilde{\Lambda}(k) \qquad \ldots(2.25)$$

and (2.25) can be satisfied with the choice

$$\left.\begin{aligned} a'_\lambda(k) &= a_\lambda(k) \qquad \lambda = 1, 2 \\ a'_3(k) &= a_3(k) - i k_0 \tilde{\Lambda}(k) \\ a'_0(k) &= a_0(k) - i k_0 \tilde{\Lambda}(k) \end{aligned}\right\} \qquad \ldots(2.26)$$

A suitable choice of $\tilde{\Lambda}(k)$, together with (2.22), now allows us to eliminate $a'_3(k)$ and $a'_0(k)$.

COULOMB GAUGE

Suppose that instead of (2.14) the following *Coulomb* condition were imposed:

$$\nabla . \underline{A} = 0 \qquad \ldots(2.27)$$

Such a constraint is of course not relativistically invariant, but it is a fortuitous choice in that (2.4a) and (2.7a) now conspire to allow us to put $A_0 = 0$, an important simplification when quantizing.

It is plain that under the Coulomb condition (2.27) it is possible to make gauge transformations (2.13) so long as the gauge function Λ satisfies an equation analogous to (2.15), namely:

$$\nabla^2 \Lambda(x) = 0 \qquad \ldots(2.28)$$

The choice of Λ which enables us to remove A_0 as well as the longitudinal part of $\underline{A}$ involves two steps[2]. Firstly, we make the transformation

$$A_\mu \to A'_\mu = A_\mu - \partial_\mu \Lambda_1 \qquad \ldots(2.29a)$$

where $$\Lambda_1(x) = \int_0^{x_0} A_0(x_0', \underline{x})\, dx_0' \qquad \ldots(2.29b)$$

and then this is followed by

$$A'_\mu \to A''_\mu = A'_\mu - \partial_\mu \Lambda_2 \qquad \ldots(2.30a)$$

where $$\Lambda_2(x) = \int \frac{d^3x}{4\pi|\underline{x}-\underline{x}'|} \nabla \cdot \underline{A}'(x_0, \underline{x}') \qquad \ldots(2.30b)$$

As a consequence of (2.4a) and (2.7a) we have

$$0 = \nabla \cdot \underline{\epsilon} = -\nabla^2 A_0 - \nabla \cdot \dot{\underline{A}} \qquad \ldots(2.31)$$

which ensures that A_0'' is zero, as is $\partial_0 \Lambda_2$.

It follows from (2.10) that $\underline{A}(x)$ satisfies the wave equation

$$\Box \; \underline{A}(x) = 0 \qquad \ldots(2.32)$$

allowing the plane wave solution

$$\underline{A}(x_0, \underline{x}) = \int \frac{d^3k}{2k_0} \underline{\epsilon}_\ell(k) \left[a_\ell(k) e^{-ik\cdot x} + a_\ell^\dagger(k) e^{ik\cdot x} \right] \qquad \ldots(2.33)$$

$$k^2 = 0 \;, \quad k_0 = |\underline{k}|$$

where ℓ takes the values 1, 2 and the sum is over ℓ is an ordinary one. In order to satisfy (2.27) it is necessary to take the unit polarization vectors $\underline{\epsilon}_\ell(k)$ orthogonal to $\underline{k}$, i.e.

$$\underline{k} \cdot \underline{\epsilon}_\ell(k) = 0 \qquad \ell = 1, 2 \qquad \ldots(2.34)$$

They may also be chosen as orthogonal to each other,

$$\underline{\epsilon}_\ell(k) \cdot \underline{\epsilon}_{\ell'}(k) = \delta_{\ell\ell'} \qquad \ldots(2.35)$$

and then $\underline{\epsilon}_1(k)$, $\underline{\epsilon}_2(k)$ and $\hat{\underline{k}} = \frac{\underline{k}}{|\underline{k}|}$ form a three-dimensional orthonormal basis system. To give this basis the same orientation under the shift $k \to -k$, it is convenient to adopt the convention

$$\left. \begin{aligned} \underline{\epsilon}_1(-k) &= -\underline{\epsilon}_1(k) \\ \underline{\epsilon}_2(-k) &= \underline{\epsilon}_2(k) \end{aligned} \right\} \qquad \ldots(2.36)$$

which implies, with (2.35), that

$$\underline{\epsilon}_{\ell}(\underline{k}) \,.\, \underline{\epsilon}_{\ell'}(-\underline{k}) = (-1)^{\ell}\,\delta_{\ell\ell'} \qquad \ldots(2.37)$$

The choice of gauge function specified in (2.29) and (2.30) is often called the *radiation gauge* since only the two transverse degrees of freedom of the wave field appear in the formalism.

So it is apparent that the use of two quite different gauge conditions leads to the same description of the potential field A_μ. This is a reflection of the fact that it is possible to perform a gauge transformation to take the Lorentz gauge into the Coulomb gauge and vice-versa. In the next section we discuss the first steps in a 'naive' quantization of this gauge field.

2.2 PRELIMINARIES TO CANONICAL QUANTIZATION

The first step in transferring to a quantum mechanical treatment of the gauge field A_μ is to find the momenta canonically conjugate to each component. These are given by the standard prescription [2, 5, 6, 7].

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial^0 A_\mu)} \qquad \ldots(2.38)$$

Using the Lagrangian of (2.12) leads to

$$\pi^0 = 0 \qquad \ldots(2.39a)$$

$$\pi^k = \partial^0 A^k - \partial^k A^0 = E^k \qquad \ldots(2.39b)$$

Note that π^k is conjugate to A_k ; i.e. $\underline{\pi}$ is conjugate to $-\underline{A}$.

The Hamiltonian density is constructed in the usual way as

$$\mathcal{H} = \pi^k \partial_0 A_k - \mathcal{L} = \tfrac{1}{2}(\underline{E}^2 + \underline{B}^2) + \underline{E}\,.\,\nabla A^0 \qquad \ldots(2.40)$$

leading to the Hamiltonian

$$H = \int d^3x\ \mathcal{H} = \tfrac{1}{2}\int d^3x\ (\underline{E}^2 + \underline{B}^2) \qquad \ldots(2.41)$$

where Gauss's law (2.4a) has been used after integration by parts.

At this point the natural field quantization procedure would require that A_μ be treated as an operator acting on a Hilbert space of possible quantum states, and that commutation relations be imposed between A_μ and its canonical momenta π^μ which correspond to the classical Poisson brackets. However, the component A_0, with its vanishing conjugate momentum, presents a problem. It is not possible to deal with A_0 in the same way as the space components A_i, thus destroying the Lorentz covariance of the theory. There are thus two distinct paths to follow:

a) Give up Lorentz covariance. By a suitable choice of gauge A_0 may be removed from the theory and sensible commutation relations imposed between the A_k and their conjugate momenta π^k. This possibility is discussed in Sections 2.4, 2.5 and 2.6.

b) Modify the Lagrangian (2.12) so as to give four non-zero components of π^μ.

The former path has the advantage that Maxwell's equations follow immediately from (2.12) and the Hamiltonian obtained, (2.41), is positive-definite. The latter path allows for a covariant quantization procedure, but leads, with a naive treatment, to a non positive-definite Hamiltonian. Further modifications are required to save the theory from unphysical results. We discuss these problems in the next section.

2.3 COVARIANT QUANTIZATION IN THE LORENTZ GAUGE

Suppose that the gauge freedom inherent in the Lagrangian (2.12) as exposed in (2.13) is used to impose the Lorentz condition (2.14) on the field A_μ. Then the Lagrangian reduces to:

$$\mathcal{L} = -\tfrac{1}{2}\, \partial_\mu A_\nu\, \partial^\mu A^\nu \qquad \text{...(2.42)}$$

This form, originally proposed by Fermi, is clearly relativistically invariant, and gauge invariant within derivatives providing the gauge function satisfies (2.15).

Now, the canonical momenta are [5, 6, 7] :

$$\pi^\mu = -\partial^0 A^\mu \qquad \text{...(2.43)}$$

The Hamiltonian density is therefore:

$$\mathcal{H} = \pi^\mu \partial_0 A_\mu - \mathcal{L} = -\tfrac{1}{2}\, \partial_0 A_\mu \partial^0 A^\mu + \tfrac{1}{2}\, \partial_i A_\mu \partial^i A^\mu \qquad \text{...(2.44)}$$

(2.44) is clearly not positive-definite, since the $\mu = 0$ component contributes a negative-definite quantity. This problem will be resolved later.

It is possible to impose the equal time commutation relations [5, 7] :

$$[\, \pi^\mu(x) \,,\, A^\nu(y) \,]_{x_0 = y_0} = i\, \eta^{\mu\nu}\, \delta\, (\underline{x} - \underline{y}) \qquad \text{...(2.45)}$$

with all other pairs of operators commuting at equal times. It follows that the unequal time commutator for the electromagnetic potentials is

$$[\, A^\mu(x),\ A^\nu(y)\,] = i\, \eta^{\mu\nu} D\,(x-y) \qquad \text{...(2.46)}$$

where $$D\,(x-y) \equiv \int d^4k\, \frac{1}{k^2 + i\epsilon}\, e^{-ik.(x-y)} \qquad \text{...(2.47)}$$

is the Fermi-Feynman propagator.

The equations of motion for operators follow as

$$i\,[\, H \,,\, A^\mu(x)\,] = \partial^0 A^\mu(x) \qquad \text{...(2.48a)}$$

$$i\,[\, H \,,\, \pi^\mu(x)\,] = \partial^0 \pi^\mu(x) \qquad \text{...(2.48b)}$$

and so on, allowing the Hamiltonian to be treated as a time translation operator.

Recalling (2.17) and (2.18), it is clear that the amplitudes $a_\mu(k)$ must be quantized according to:

$$\left.\begin{aligned} [\, a_\mu(k),\ a_\nu^+(k)\,] &= -2k_0\, \eta_{\mu\nu}\, \delta\,(\underline{k} - \underline{k}') \\ [\, a_\mu(k),\ a_\nu(k)\,] &= [\, a_\mu^+(k),\ a_\nu^+(k)\,] = 0 \end{aligned}\right\} \qquad \text{...(2.49)}$$

leading to the Hamiltonian

$$H = -\frac{1}{2} \int d^3k \; a_\mu^\dagger(k)\, a^\mu(k) \qquad \ldots(2.50)$$

A representation is now required for the operators $a_\mu(k)$ and $a_\mu^\dagger(k)$. There is the possibility:

INTERPRETATION 1 $a_\mu(k)$ destruction operators

$a_\mu^\dagger(k)$ creation operators.

Here the vacuum is characterized by

$$a_\mu(k)\,|0\rangle = 0 \quad \forall\, k, \mu \qquad \ldots(2.51a)$$

or equivalently, by

$$A_\mu^{(+)}(k)\;|0\rangle = 0 \quad \forall\, x, \mu \qquad \ldots(2.51b)$$

where $A_\mu^{(+)}(x)$ is the destruction part of the operator $A_\mu(x)$. But this leads to the following problem [5, 7] :

PROBLEM 1

There exist states with negative norm. For we have

$$\langle 0|\, a_0(k)\, a_0^\dagger(k)\, |0\rangle = -2k_0\,\delta(\underline{k}-\underline{k}') \qquad \ldots(2.52)$$

So the state $\int f(\underline{k})\, a_0^\dagger(k)\, d\underline{k}|0\rangle$ with $\int |f(\underline{k})|^2 d\underline{k} < \infty$ has negative norm. This problem is of course of a purely formal nature since time-like photons play no role in determining observable quantities, but nevertheless we could try and overcome it with another representation:

INTERPRETATION 2 $a_\ell(k)$, $a_0^\dagger(k)$ destruction operators

$a_\ell^\dagger(k)$, $a_0(k)$ creation operators

Unfortunately, although Problem 1 is now disposed of, there is still a difficulty.

PROBLEM 2

States of arbitrarily high negative energy exist. For instance, $a_0(k)|0\rangle$, the state containing one timelike photon, has

$$H\; a_0(k)\,|0\rangle = -k_0\, a_0(k)\,|0\rangle \qquad \ldots(2.53)$$

In any case, regardless of which interpretation is chosen, there is also:

PROBLEM 3

The Lorentz condition (2.14) cannot be imposed as an operator identity for it would contradict (2.46). That is:

$$[\partial^\mu A_\mu(x), A_\nu(y)] = -i\,\partial_\nu D(x-y) \qquad (\neq 0) \qquad \ldots(2.54)$$

Thus the theory does not as yet correspond to the Maxwell theory, insofar as $\langle A_\mu(x)\rangle_\psi \equiv \langle\psi|A_\mu(x)|\psi\rangle$ does not satisfy $\partial^\mu \langle A_\mu(x)\rangle_\psi = 0$ and so Maxwell's equations are not satisfied by $\langle F_{\mu\nu}(x)\rangle_\psi$.

One way to circumvent Problem 3 is to introduce a scalar Stuckelberg field into the formalism [6]. Another method which is more often cited is due to Gupta and Bleuler [1, 5, 4, 6, 7]. We will present a short outline of the latter method here.

GUPTA-BLEULER METHOD

Interpretation 2 is adopted, and the vector space is modified by the introduction of an 'indefinite metric'. The scalar product which makes the space of states into a Hilbert space [7] is abandoned in favour of a new 'Gupta' scalar product which is related to the old one by

$$(\Psi, \chi)_G = (\Psi, \eta\,\chi) \qquad \ldots(2.55)$$

where η is a linear hermitian operator called the 'metric operator' and taken to satisfy

$$\begin{aligned} \eta^2 &= 1 \\ \eta &= (-1)^{n_0} \end{aligned} \qquad \ldots(2.56)$$

where n_0 is the number of timelike photons in the state χ. Note that (2.55) ensures that $a_\mu^+(k)$ is the adjoint of $a_\mu(k)$ under Interpretation 2. One could proceed by giving a Fock space treatment of the creation and destruction operators, but we are merely content here to state some results without proof.

Firstly, it should be noted that

$$\partial^\mu A_\mu(x)\,|\Psi\rangle = 0 \quad \forall\, x \qquad \ldots(2.57)$$

is too strict a condition when $A_\mu(x)$ is an operator on states, for it requires not only that certain kinds of photons not be present, but also that they cannot be emitted. After all, to derive Maxwell's equations all that is needed is

$$(\Psi, \partial^\mu A_\mu \Psi) = 0 \qquad \ldots(2.58)$$

and this will hold if

$$\partial^\mu A_\mu^{(+)}(x)\,|\Psi\rangle = 0 \quad \forall x \qquad \ldots(2.59)$$

That is, the subsidiary condition analogous to (2.14) is now

$$L(k)\,|\Psi\rangle \equiv k^\mu a_\mu(k)\,|\Psi\rangle = 0 \quad \forall k \qquad \ldots(2.60)$$

Since $A_\mu(x)$ satisfies (2.16), its decomposition into positive and negative frequencies is relativistically invariant.

There are now some straightforward results [7]:

1) Denoting by $\mathcal{P}$ the manifold of states $|\Psi\rangle$ satisfying the subsidiary condition (2.60), any photons contained in a state $|\Psi\rangle \in \mathcal{P}$ are transversely polarized. The proof is given in ref. [7] for one-photon states, and is similar to the Lorentz gauge discussion in sec. 2.1 which removes the two unwanted components of the field.

2) A state is called *physically realizable* if it satisfies (2.60) and contains only transverse photons. It follows that physically realizable states have a positive-definite norm within the Gupta scalar product.

i.e. $$(\Psi, \Psi)_G > 0 \qquad \ldots(2.61)$$

3) The energy-momentum four-vector is positive-semidefinite in $\mathcal{P}$. This is easily seen once it is realized that a transformation can always be made to the equivalence class of states containing an equal number of longitudinal and timelike photons of the same momentum. The negative energy (momentum) contribution of the timelike photons cancels that of the longitudinal photons.

One could choose the state $|\chi\rangle$ containing no timelike or longitudinal photons as the representative of the equivalence class. This corresponds to picking a particular gauge.

The main feature of the Gupta-Bleuler formalism is that states with timelike and longitudinal photons are included and well defined, but nonetheless do not contribute to the physical theory.

2.4 COULOMB GAUGE QUANTIZATION

There is a natural method of quantization which avoids all the problems of the preceding section, but at the cost of manifest Lorentz covariance. In this procedure, described in ref. [2], the Lagrangian (2.12) is retained, and the canonical momenta are given in (2.39), leading to the Hamiltonian (2.41). The only non-zero equal time commutation relation which the canonical procedure then imposes is

$$[\pi^i(x), A_j(y)]_{x_0=y_0} = -[E^i(x), A_j(y)]_{x_0=y_0} = -i\,\delta_{ij}\,\delta(\underline{x}-\underline{y}) \quad \ldots(2.62)$$

Clearly, (2.62) is incompatible with Gauss's law (2.4a), for

$$\partial^i \delta_{ij}\,\delta(\underline{x}-\underline{y}) = i\int d^3k\; e^{i\underline{k}\cdot(\underline{x}-\underline{y})}\,k_j \quad (\neq 0) \quad \ldots(2.63)$$

This problem is easily overcome if the δ-function is modified to

$$\delta^{tr}_{ij}(\underline{x}-\underline{y}) = \int d^3k\; e^{i\underline{k}\cdot(\underline{x}-\underline{y})}\left(\delta_{ij} - \frac{k_i k_j}{|\underline{k}|^2}\right) \quad \ldots(2.64)$$

leading to a new commutation relation

$$[\pi^i(x), A^j(y)]_{x_0=y_0} = i\,\delta^{tr}_{ij}(\underline{x}-\underline{y}) \quad \ldots(2.65)$$

There is no need for alarm at the change (2.64), for all that has been done is to require that (2.27) apply, a condition which is reflected in (2.65). Under this Coulomb gauge quantization scheme, $\nabla . \underline{A}$ is a c-number commuting with all operators, and so following the discussion in section 2.1 both it and A_0 have been eliminated as dynamical variables. The gauge transformation (2.29b), (2.30b) can now be used to remove them from the theory.

As demonstrated in ref. [2], this quantization is covariant in the sense that the theory is invariant under coordinate displacements and spatial rotations. Under a Lorentz transformation, a gauge term is necessary to maintain the covariance of Maxwell's equations.

In the plane wave form for $\underline{A}(x)$ of (2.33), the amplitudes $a_m(\underline{k})$, $a_m^{\dagger}(\underline{k})$ defined through (2.19) are interpreted as destruction and creation operators respectively. Their non-zero equal time commutation relation is

$$[\, a_m(\underline{k}) \, , \, a_n^{\dagger}(\underline{k}') \,] = 2 k_0 \delta_{mn} \delta (\underline{k} - \underline{k}') \qquad \ldots (2.66)$$

leading to the Hamiltonian

$$H = \frac{1}{2} \int d^3k \; a_m^{\dagger}(\underline{k}) \, a_m(\underline{k}) \qquad \ldots (2.67)$$

There are no problems in this gauge with the positive-definiteness of (2.67), as is guaranteed through (2.41).

2.5 TEMPORAL GAUGE QUANTIZATION

The temporal gauge has the characteristic that the canonical Hilbert space generated by the quantization procedure is larger than the set of states of physical interest. By applying Gauss's law (2.4a) as a condition, unphysical states are eliminated.

The temporal gauge condition [8, 9, 10, 11]

$$A_o = 0 \qquad \ldots (2.68)$$

is imposed on the Lagrangian (2.12). Removing the time component at this stage precludes its elimination again as a dependent variable. Condition (2.68) still allows a restricted class of gauge transformations, analogous to (2.15) and (2.29) for the Lorentz and Coulomb gauges respectively. That is, one can make the change

$$\underline{A}(x_0,\underline{x}) \rightarrow \underline{A}'(x_0,\underline{x}) = \underline{A}(x_0,\underline{x}) + \nabla\Lambda(\underline{x}) \qquad \ldots(2.69)$$

where $\Lambda(\underline{x})$ is an arbitrary function of the space coordinates only.

The dynamical coordinates left after (2.68) is applied are the space components A_i $(i = 1,2,3)$, leading to the canonical momenta

$$\pi^i = \frac{\partial\mathcal{L}}{\partial(\partial^0 A_i)} = F^{0i} = \partial^0 A^i = E^i \qquad \ldots(2.70)$$

The Hamiltonian is the usual one, (2.41). The equal time commutation relations which are now imposed are the same as for the Coulomb gauge, i.e. (2.62). Thus the equations of motion of the quantum operators which follow from commuting them with the Hamiltonian are:

$$\begin{aligned}\partial^0 A^i(x) &= i\,[H, A^i(x)]\\ &= E^i(x) \qquad \ldots(2.71a)\end{aligned}$$

and

$$\begin{aligned}\partial^0 E^i(x) &= i\,[H, E^i(x)]\\ &= \partial^j F^{ji}(x)\\ &= (\nabla\times\underline{B})^i(x) \qquad \ldots(2.71b)\end{aligned}$$

Unfortunately, Gauss's law (2.4a) does not follow from these quantum mechanical equations of motion. This spells trouble for a consistent quantum treatment of the dynamical variables. But from (2.71b) we do have at least that

$$\partial_0(\nabla.\underline{E}) = 0 = i\,[H, \nabla.\underline{E}] \qquad \ldots(2.72)$$

So H and $\nabla\cdot\underline{E}$ commute, and can be formally simultaneously diagonalised; i.e. they share the same eigenstates. Because of this, it seems reasonable to propose that a state $|\Psi\rangle$ should be called 'physical' if it satisfies

$$\nabla\cdot\underline{E}\;|\Psi\rangle = 0 \qquad \text{...(2.73)}$$

This physicality condition (2.73) is clearly incompatible with the commutation relations (2.62), since

$$[\,\nabla\cdot\underline{E}(x),\, A^j(y)]_{x_0=y_0} = i\,\partial^i\,\delta_{ij}\,\delta(\underline{x}-\underline{y}) \quad (\neq 0) \qquad \text{...(2.74)}$$

In particular, for an arbitrary state $|\Psi\rangle$,

$$\langle\Psi|[\,\nabla\cdot\underline{E}(x),\, A^j(y)]_{x_0=y_0}|\Psi\rangle = i\,\partial^j\,\delta(\underline{x}-\underline{y})\langle\Psi|\Psi\rangle \qquad \text{...(2.75)}$$

and if (2.74) is insisted on, it becomes impossible to normalize the eigenstates of $\nabla\cdot\underline{E}$ since $\langle\Psi|\Psi\rangle$ must be zero. To solve this problem, a limiting procedure is applied on states where $\nabla\cdot\underline{E}$ is smeared slightly about zero [8, 9]. The smearing process is discussed in ref. [12] and will not be treated here. However, a brief account of the limiting procedure is now given.

The momentum space representations of the operators $\underline{A}$ and $\underline{E}$ at $x^0=0$ analogous to (2.17) are

$$A_i(\underline{x}) = \int\frac{d^3k}{2k_0}\,\epsilon_{i\ell}(\underline{k})\left[a_\ell(\underline{k})e^{i\underline{k}\cdot\underline{x}} + a_\ell^{\dagger}(\underline{k})\,e^{-i\underline{k}\cdot\underline{x}}\right] \qquad \text{...(2.76a)}$$

$$E_i(\underline{x}) = -i\int\frac{d^3k\,k_0}{2}\,\epsilon_{i\ell}^{-1}(\underline{k})\left[a_\ell(\underline{k})e^{i\underline{k}\cdot\underline{x}} - a_\ell^{\dagger}(\underline{k})\,e^{-i\underline{k}\cdot\underline{x}}\right] \qquad \text{...(2.76b)}$$

$$(\ell = 1,2,3)$$

where the amplitudes are quantized according to

$$[\,a_\ell(\underline{k}),\, a_{\ell'}^{\dagger}(\underline{k}')\,] = 2k_0\,\delta_{\ell\ell'}\,\delta(\underline{k}-\underline{k}') \qquad \text{...(2.77)}$$

To eliminate the unwanted longitudinal degrees of freedom, a special choice of polarization matrix is made. Defining the transverse projection operator by

$$P_{i\ell} = \delta_{i\ell} - \frac{k_i k_\ell}{|\underline{k}|^2} \qquad \text{...(2.78)}$$

the polarization matrix is chosen as

$$\begin{aligned}\epsilon_{i\ell}(\underline{k}) &= P_{i\ell} + \frac{1}{\alpha}(1-P)_{i\ell}\\ &= \delta_{i\ell} - \frac{1-\alpha}{\alpha}\,\frac{k_i k_\ell}{|\underline{k}|^2}\end{aligned} \qquad \text{...(2.79)}$$

where α is an arbitrary constant. And so it follows that

$$\epsilon^{-1}_{i\ell}(\underline{k}) = P_{i\ell} + \alpha(1-P)_{i\ell}$$
$$= \delta_{i\ell} + (\alpha-1)\frac{k_i k_\ell}{|\underline{k}|^2} \qquad \ldots(2.80)$$

Eqs. (2.76b), (2.79) and (2.80) now give

$$\nabla.\underline{E}(x) = \alpha \int \frac{d^3k\,k_0}{2}\left[\underline{k}.\underline{a}(\underline{k}) - \underline{k}.\underline{a}^+(-\underline{k})\right]e^{i\underline{k}.\underline{x}} \qquad \ldots(2.81)$$

Thus Gauss's law (2.4a) is recovered in the limit as $\alpha \to 0$. In ref. [8] the states are made to carry the α-dependence, and a Fock space representation is given to them with a restricted Lorentz condition being imposed in an analogous way to (2.60). It should be noted that in any calculations the limit of vanishing α cannot be taken until an expression has been obtained in terms of physical quantities only.

Of interest is the form of the Hamiltonian, which is

$$H = \tfrac{1}{2}\int d^3x\,[\underline{E}(x) + (\nabla\times\underline{A})(x)]$$
$$= \tfrac{1}{8}\int d^3k\,\Big\{4\,\underline{a}^+(\underline{k}).\underline{a}(\underline{k})$$
$$- \frac{\alpha^2}{k_0^2}\left[\underline{k}.\underline{a}(\underline{k})\,\underline{k}.\underline{a}(-\underline{k}) - 2\,\underline{k}.\underline{a}^+(\underline{k})\,\underline{k}.\underline{a}(\underline{k}) + \underline{k}.\underline{a}^+(\underline{k})\,\underline{k}.\underline{a}^+(-\underline{k})\right]\Big\} \qquad \ldots(2.82)$$

Thus, in the limit $\alpha \to 0$, (2.82) reduces to (2.67).

One further point should be made. The restricted gauge transformations of (2.69) are generated by the unitary operator

$$U = \exp\left\{-i\int d^3x\,\Lambda(\underline{x})\,\nabla.\underline{E}(x)\right\} \qquad \ldots(2.83)$$

Because $\nabla.\underline{E}$ vanishes on physical states, it follows that

$$\langle \Psi', \alpha | U | \Psi, \alpha \rangle = \langle \Psi', \alpha | \Psi, \alpha \rangle + O(\alpha^2) \qquad \ldots(2.84)$$

so that physical states are symmetric under local gauge transformations. Thus (2.83) and (2.84) guarantee that gauge invariant operators will respect Gauss's law and take physical states into physical states.

2.6 AXIAL GAUGE QUANTIZATION

Consider the following axial gauge condition [13]:

$$n^\mu A_\mu = 0 \qquad \ldots(2.85)$$

where n^μ is an arbitrary four-vector. (2.85) is imposed on the Lagrangian (2.12), and can be viewed as a constraint which eliminates one of the components of the A-field. The temporal gauge is a particular case.

In the following, equations which arise involving the field observables $F_{\mu\nu}$ will be taken as ones either of constraint or of motion, and used accordingly. We will also use a notation which decomposes any four-vector V_μ into transverse and longitudinal components with respect to n_μ, viz:

$$V_\mu = V_{T\mu} + V_{L\mu} \qquad \text{...(2.86)}$$

where $$n^\nu V_{T\nu} = 0 \qquad \text{...(2.87a)}$$

and $$V_{L\mu} = \frac{n^\nu V_\nu}{n^2} n_\mu \qquad \text{...(2.87b)}$$

The canonical momenta conjugate to each A are:

$$\pi^i = \frac{\partial \mathcal{L}}{\partial(\partial A_i)} = F^{oi} = E^i \qquad \text{...(2.88)}$$

From the definition (2.8) and Maxwell's equations (2.6) there are the following equations of motion:

$$\partial^o A_i = \partial_i A_o + E_i \qquad \text{...(2.89a)}$$

and $$\partial^o E_i = \partial^j F_{ij} \qquad \text{...(2.89b)}$$

and constraint equations:

$$\partial_i F_{oi} = 0 \qquad \text{(i.e. Gauss's law)} \qquad \text{...(2.90a)}$$

and $$F_{ij} = \partial_i A_j - \partial_j A_i \qquad \text{...(2.90b)}$$

If (2.89a) is multiplied by n_i we get, using (2.85),

$$(n^\nu \partial_\nu) A_o = n_i E_i \qquad \text{...(2.91)}$$

which allows A_o to be eliminated as a dynamical variable.

Now we take a brief look at some different choices of n_μ.

n_μ *SPACELIKE*

Suppose

$$n_\mu = (o, \underline{n}) \qquad \text{...(2.92)}$$

Then the gauge condition (2.85) becomes

$$\underline{n} \cdot \underline{A} = 0 \qquad \text{...(2.93)}$$

and (2.91) is now

$$\underline{n} \cdot \underline{\partial}\, A_0 = \underline{n} \cdot \underline{E} \qquad \ldots(2.94)$$

A solution to (2.94) is [13]:

$$A_0\,(x_T, x_L) = \int_{-\infty}^{x_L} dx_L'\, E_L\,(x_T, x_L') \qquad \ldots(2.95)$$

where $x_L = \frac{\underline{x}\cdot\underline{n}}{|\underline{n}|}$ from (2.87b).

To find E_L, consider (2.90a). Since ∂_i can be written as

$$\partial_i = \partial_{Ti} + \partial_L \frac{n_i}{|\underline{n}|} \qquad \ldots(2.96)$$

we have

$$\partial_L E_L = -\partial_{Ti} E_i \qquad \ldots(2.97)$$

which yields the solution

$$E_L(x_T, x_L) = -\int_{-\infty}^{x_L} dx_L'\, \partial_{Ti} E_{Ti}(x_T, x_L') \qquad \ldots(2.98)$$

So the only independent variable remaining is E_T.

The equal-time commutation relations which are now imposed between the operators are

$$[\, E_{Ti}(x)\,, A_{Tj}(y)\,] = -i\, \delta_{Tij}\, \delta\,(\underline{x} - \underline{y}) \qquad \ldots(2.99)$$

where δ_{Tij} is defined in an analagous way to (2.64) as

$$\delta_{Tij} \equiv \delta_{ij} - \frac{n_i n_j}{|\underline{n}|} \qquad \ldots(2.100)$$

The Hamiltonian is

$$H = \tfrac{1}{2} \int d^3x\, [\, E_{Ti} E_{Ti} + E_L E_L + \tfrac{1}{2} F_{ij} F_{ij}\,] \qquad \ldots(2.101)$$

SPATIAL GAUGE

If n_μ is chosen to be

$$n_\mu = (0, 0, 0, 1) \qquad \ldots(2.102)$$

then the gauge condition (2.85) becomes

$$A_3 = 0 \qquad \ldots(2.103)$$

This was in fact the first type of axial gauge given serious consideration [14, 15, 16].

The equations of motion and constraint are unchanged from (2.89) and (2.90), with the proviso that i and j now run only over the values 1 and 2. The corresponding equation to (2.94) which will eliminate A_0 from dynamical considerations is just

$$\partial_3 A_0 = -E_3 \qquad \ldots(2.104)$$

and one could continue in the same vein as the previous discussion.

TEMPORAL GAUGE

Obviously, if n_μ is taken as

$$n_\mu = (1, 0, 0, 0) \qquad \ldots(2.105)$$

we encounter just the temporal gauge of the last section.

'NULL-PLANE' GAUGE:

If $n^2 = 0$, it is convenient to take [17].

$$n_\mu = \tfrac{1}{\sqrt{2}}(1, 0, 0, 1) \qquad \ldots(2.106)$$

and introduce 'null coordinates'

$$\left.\begin{aligned} x_\pm &= \tfrac{1}{\sqrt{2}}(x_0 \pm x_3) \\ \underline{x} &= (x_1, x_2) \end{aligned}\right\} \qquad \ldots(2.107)$$

and a 'null metric'

$$\left.\begin{aligned} \eta_{+-} &= \eta_{-+} = 1 \\ \eta_{++} &= \eta_{--} = 0 \\ \eta_{ij} &= -\delta_{ij} \; ; \; i, j = 1, 2 \end{aligned}\right\} \qquad \ldots(2.108)$$

The gauge condition following from (2.106) and (2.85) is thus

$$A_- = 0 \qquad \ldots(2.109)$$

leading to the field equations

$$\partial^k F_{+k} = 0 \qquad k = 1, 2, - \qquad \ldots(2.110a)$$

$$\partial^\mu F_{\mu i} = 0 \qquad i = 1, 2 \qquad \ldots(2.110b)$$

(2.110b) is a constraint equation which may be solved to eliminate A_+:

i.e.
$$(\partial_-)^2 A_+ = -\partial^i F_{+i} \qquad \ldots(2.111)$$

So only A_1 and A_2 are left as dynamical variables.

So we have had a glance at the application of the canonical quantization procedure in four gauge choices. It is possible to proceed by this method to the Feynman propagator in each gauge, but this is more simply obtained via the path integral technique, which is discussed in the next chapter.

* * * * * * * * * * * *

REFERENCES - CHAPTER 2

1. A.I. Akheizer and V.B. Berestetskii : *'Quantum Electrodynamics'*, Interscience Publishers, N.Y. (1965).

2. J.D. Bjorken and S.D. Drell : *'Relativistic Quantum Field Theory'*, McGraw-Hill, U.S.A. (1964).

3. N.N. Boguliubov and D.V. Shirkov : *'Introduction to the theory of quantized fields'*, Interscience Publishers, N.Y. (1959).

4. J.M. Jauch and F. Rohrlich : *'The Theory of Photons and Electrons'*, Springer-Verlag, N.Y. (1976).

5. S. Gasorowicz : *'Elementary Particle Physics'*, Wiley and Sons, N.Y. (1966).

6. K. Nishijima : *'Fields and Particles'*, W.A. Benjamin, N.Y. (1969).

7. S.S. Schweber : *'An Introduction to Relativistic Quantum Field Theory'*, Harper and Row, N.Y. (1962).

8. M. Creutz : Ann. Phys. 117, 471 (1979).

9. M. Creutz, I.J. Muzinich and T.N. Tudron : Phys. Rev. D19, 531 (1979).

10. M. Creutz and T.N. Tudron : Phys. Rev. D16, 2978 (1977).

11. M. Creutz and T.N. Tudron : Phys. Rev. D17, 2619 (1978).

12. J.F. Willemson : Phys. Rev. D17, 574 (1978).

13. J. Frenkel : Nuovo Cimento A32, 313 (1976).

14. R.L. Arnowitt and S.I. Fickler : Phys. Rev. 127, 1821 (1962).

15. J. Schwinger : Phys. Rev. 130, 402 (1963).

16. York-Peng Yao : J. Math. Phys. 5, 1319 (1964).

17. E. Tomboulis : Phys. Rev. D8, 2736 (1973).

3. BACKGROUND FORMALISM

Before outlining the calculations of the renormalization constants, it is useful to go over several topics to do with the quantization of non-abelian gauge theory. This chapter is not meant as a review or a mathematical treatise. Rather, here will be found the framework on which the calculations are based. Feynman rules will be given for various gauges; renormalization constants defined and relations between them due to the gauge invariance derived; and the usefulness of dimensional regularization discussed.

3.1 NON-ABELIAN GAUGE THEORY

In Chapter 2 we saw how to canonically quantize the free Maxwell field, taking into account its gauge invariance. In this section, we extend the theory to include charged scalar mesons and spin-1/2 fermions, and then generalise it to allow non-abelian gauge transformations. The Lagrangian for such a theory is given by minimal substitution as

$$\begin{aligned}\mathcal{L} &= -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + \phi^{+}(\overleftarrow{\partial}_{\mu} + ieA_{\mu})(\overrightarrow{\partial}^{\mu} - ieA^{\mu})\phi - M^{2}\phi^{+}\phi \\ &\quad + \bar{\psi}\,\gamma^{\mu}(i\,\partial_{\mu} + eA_{\mu})\psi - m\bar{\psi}\psi \\ &= -\tfrac{1}{4}(\partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu})(\partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}) + \phi^{+}(\overleftarrow{\partial}_{\mu}\overrightarrow{\partial}^{\mu} - M^{2})\phi \\ &\quad + ie\phi^{+}A_{\mu}(\partial^{\mu}\phi) - ie(\partial_{\mu}\phi^{+})A^{\mu}\phi + e^{2}\,\phi^{+}A_{\mu}\phi A^{\mu} \\ &\quad + \bar{\psi}\,(i\not{\partial} - m)\psi + e\bar{\psi}\not{A}\psi \qquad \ldots(3.1)\end{aligned}$$

where ϕ is the scalar meson field

ψ is the fermion field

and e is the coupling constant between these and the gauge field. The quartic $(\phi^{+}\phi)^{2}$ contribution to (3.1) has been omitted since in Chapters 4 and 5 we will only be concerned with second order calculations in the gauge field coupling constant, so that this term is merely an unnecessary complication at this stage.

Such a Lagrangian is invariant under the following abelian local gauge transformations [1,2,3,4,5,6]:

$$\left.\begin{aligned}
A_\mu(x) &\to A_{\mu\Lambda}(x) \equiv A_\mu(x) - \partial_\mu \Lambda(x) \\
\phi(x) &\to \phi_\Lambda(x) \equiv \exp[ie\Lambda(x)]\phi(x) \simeq [1+ie\Lambda(x)]\phi(x) \\
\phi^+(x) &\to \phi^+_\Lambda(x) \equiv \phi^+(x)\exp[-ie\Lambda(x)] \simeq \phi^+(x)[1-ie\Lambda(x)] \\
\psi(x) &\to \psi_\Lambda(x) \equiv \exp[ie\Lambda(x)]\psi(x) \simeq [1+ie\Lambda(x)]\psi(x) \\
\bar\psi(x) &\to \bar\psi_\Lambda(x) \equiv \bar\psi(x)\exp[-ie\Lambda(x)] \simeq \bar\psi(x)[1-ie\Lambda(x)]
\end{aligned}\right\} \quad \ldots(3.2)$$

where $\Lambda(x)$ is a gauge function belonging to the group $U(1)$.

It is natural to define *covariant derivatives*

$$\left.\begin{aligned}
D_\mu A_\nu &\equiv \partial_\mu A_\nu \\
D_\mu \phi &\equiv (\partial_\mu - ieA_\mu)\phi \\
D_\mu \psi &\equiv (\partial_\mu - ieA_\mu)\psi
\end{aligned}\right\} \quad \ldots(3.3)$$

and using these the equations of motion are

$$\left.\begin{aligned}
\partial_\mu F^{\mu\nu} &= ie\phi^+ \overleftrightarrow{D}^\nu \phi + e\bar\psi\gamma^\nu\psi \\
(D_\mu D^\mu - M^2)\,\phi &= 0 \\
(i\not{D} - m)\,\psi &= 0
\end{aligned}\right\} \quad \ldots(3.4)$$

It is possible to generalize the gauge group to $SU(n)$ by introducing matrix representatives T^a of the group generators so that $\Lambda(x)$ becomes $T^a\Lambda^a(x)$[2,6,7]. The matrices obey the commutation relation

$$[T^a, T^b] = i f^{abc} T^c \quad \ldots(3.5)$$

where the f^{abc} are the structure constants for the group. Further properties are discussed in Appendix B.

In order to retain a gauge invariant Lagrangian it is necessary to replace the electromagnetic field strength $F_{\mu\nu}$ by the *'Yang-Mills' field* [8], defined by:

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \quad \ldots(3.6)$$

where A^a_μ is an isovector with n components labelled by the group index a. (Sometimes we will write $\underline{A}_\mu \times \underline{A}_\nu$ to represent

$f^{abc}A_\mu^b A_\nu^c$). Thus, the Lagrangian for the new theory is

$$\begin{aligned}\mathcal{L} &= -\tfrac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} + \phi^+(\overleftarrow{\partial}_\mu + igT^aA_\mu^a)(\overrightarrow{\partial}^\mu - igT^bA_\mu^b)\phi \\ &\quad - M^2\phi^+\phi + \bar\psi\gamma^\mu(i\partial_\mu + gT^aA_\mu^a)\psi - m\bar\psi\psi \\ &= -\tfrac{1}{4}[\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc}A_\mu^bA_\nu^c][\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu} + gf^{abc}A^{b\mu}A^{c\nu}] \\ &\quad + \phi^+(\overleftarrow{\partial}_\mu\overrightarrow{\partial}^\mu - M^2)\phi + igT^a\phi^+A_\mu^a\partial^\mu\phi - ig\partial_\mu\phi^+T^a\phi A^{a\mu} \\ &\quad + g^2T^a\phi^+A_\mu^aT^b\phi A^{b\mu} + \bar\psi(i\partial\!\!\!/ - m)\psi + g A\!\!\!/^a\bar\psi T^a\psi\end{aligned} \qquad \ldots(3.7)$$

where g is the coupling constant, and the quartic $(\phi^+\phi)^2$ term is omitted as in (3.1).

The Lagrangian (3.7) is invariant under the following non-abelian local gauge transformations [6,7]:

$$\left.\begin{aligned} A_\mu^a(x) &\to A_{\mu\Lambda}^a(x) \equiv A_\mu^a(x) - \partial_\mu\Lambda^a(x) + gf^{abc}\Lambda^b(x)A_\mu^c(x) \\ \phi(x) &\to \phi_\Lambda(x) \equiv \exp[igT^a\Lambda^a(x)]\phi(x) \simeq [1 + igT^a\Lambda^a(x)]\phi(x) \\ \phi^+(x) &\to \phi_\Lambda^+(x) \equiv \phi^+(x)\exp[-igT^a\Lambda^a(x)] \simeq \phi^+(x)[1 - igT^a\Lambda^a(x)] \\ \psi(x) &\to \psi_\Lambda(x) \equiv \exp[igT^a\Lambda^a(x)]\psi(x) \simeq [1 + igT^a\Lambda^a(x)]\psi(x) \\ \bar\psi(x) &\to \bar\psi_\Lambda(x) \equiv \bar\psi(x)\exp[-igT^a\Lambda^a(x)] \simeq \bar\psi(x)[1 - igT^a\Lambda^a(x)] \end{aligned}\right\} \quad \ldots(3.8)$$

where $T^a\Lambda^a(x)$ belongs to the gauge group $SU(n)$.

Defining new *covariant derivatives* by

$$\left.\begin{aligned} D_\mu A_\nu^a &\equiv \partial_\mu A_\nu^a + \tfrac{1}{2}gf^{abc}A_\mu^bA_\nu^c \\ D_\mu\phi &\equiv (\partial_\mu - igT^aA_\mu^a)\phi \\ D_\mu\psi &\equiv (\partial_\mu - igT^aA_\mu^a)\psi \end{aligned}\right\} \quad \ldots(3.9)$$

allows the equations of motion to be written as

$$\left.\begin{aligned} \partial_\mu F^{\mu\nu} &= -gf^{abc}A_\mu^bF^{c\mu\nu} + igT^a\phi^+D^\nu\phi + g\bar\psi\gamma^\nu T^a\psi \\ (D_\mu D^\mu - M^2)\phi &= 0 \\ (i\partial\!\!\!/ - m)\psi &= 0 \end{aligned}\right\} \quad \ldots(3.10)$$

Appearing on the left hand side of the first equation in both sets (3.4) and (3.10) we find the term (forgetting group indices for the moment):

$$\partial_\mu F^{\mu\nu} = (\partial^2\eta_{\mu\nu} - \partial_\mu\partial_\nu)A^\nu \qquad \ldots(3.11)$$

Now the factor $(\partial^2\eta_{\mu\nu} - \partial_\mu\partial_\nu)$ is singular, and therefore cannot be inverted to obtain the propagator in a quantized theory.

This is associated with the gauge invariance of the Lagrangians (3.1) and (3.7). To find the propagator for the gauge field it is therefore necessary to settle upon a gauge condition in some way. One method is to add a gauge fixing term to the Lagrangian which incorporates the condition.

The choice of gauge may involve placing any scalar condition on the potential field, in general described by

$$F[A_\mu] = B \qquad \ldots(3.12)$$

In the next section we discuss in general terms how this condition may be imposed consistently in the path-integral quantization of non-abelian gauge theory.

3.2 PATH-INTEGRAL QUANTIZATION

Functional methods [8,9,10,11] allow a 'covariant' quantization procedure in which the Green's functions for a theory are derived via the path integral technique [2,7,12, 13,14,15,16,17,18,19,20]. In non-abelian gauge theory this procedure is especially simple and useful for exhibiting identities associated with the gauge invariance. The details of this technique can be found in the above references. Here we merely quote results applicable to the work at hand.

We introduce source terms j_μ^a, j and η for the gauge, scalar meson and fermion fields respectively. Then the generating functional Z is given by the functional integral

$$Z[j_\mu^a, j, j^+, \eta, \bar{\eta}] = \int [dA_\mu^a \, d\phi \, d\phi^+ d\psi \, d\bar{\psi}] \times$$
$$\times \exp i \int (\mathcal{L} - j^a A_\mu^a - j\phi^+ - j^+\phi - \eta\bar{\psi} - \bar{\eta}\psi) \qquad \ldots(3.13)$$

where $\mathcal{L}$ is given in (3.7). Here we have used the abbreviations:

$$\left.\begin{aligned} \int \mathcal{L} &\equiv \int \mathcal{L}[A_\mu^a(x), \partial^\mu A_\mu^a(x), \phi(x), \psi(x)] d^4x \\ \int [dA_\mu^a] &\equiv \lim_{n\to\infty} \prod_n \int dA_{\mu_n}^a(x_n) \end{aligned}\right\} \qquad \ldots(3.14)$$

Also, in the following we will often drop group indices where they shed no particular light on the discussion.

The Green's functions are obtained as source functional derivatives of the generating functional Z. For example,

$$\langle T A_{\mu_1}(x_1)\ldots A_{\mu_n}(x_n)\rangle = \int [dA_\mu\ldots] A_{\mu_1}(x_1)\ldots A_{\mu_n}(x_n) \exp i\int(\mathcal{L}+\cdots)$$
$$= i^n \frac{\delta^n Z[j_\mu\ldots]}{\delta j_{\mu_1}(x_1)\ldots\delta j_{\mu_n}(x_n)} \qquad \ldots(3.15)$$

It follows that the physical Green's functions in the absence of external sources are

$$G(x_1,\mu_1;\ldots;x_n,\mu_n) = i^n \left.\frac{\delta^n Z[j_\mu\ldots]}{\delta j_{\mu_1}(x_1)\ldots\delta j_{\mu_n}(x_n)}\right|_{j_{\mu_i}=0} \qquad \ldots(3.16)$$

and so on for other fields.

The Euler-Lagrange equations of motion follow as source functional derivatives on Z. For example,

$$\begin{aligned} 0 &= \int [dA_\mu\ldots] \frac{\delta}{\delta A_\mu(x)} \exp i\int(\mathcal{L}-j^\mu A_\mu+\cdots) \\ &= \int [dA_\mu\ldots] \left(\frac{\partial\mathcal{L}}{\partial A_\mu(x)} - \partial_\nu \frac{\partial\mathcal{L}}{\partial(\partial_\nu A_\mu(x))} - j_\mu(x)\right)\exp i\int(\mathcal{L}-j^\mu A_\mu+\cdots) \\ &= \left(\mathcal{L}'\left[i\frac{\delta}{\delta j_\mu(x)}\right] - j_\mu(x)\right) Z[j_\mu\ldots] \end{aligned} \qquad \ldots(3.17)$$

Perturbation expansions come from expanding the non-quadratic interactions as a power series in the coupling constant.

Connected Green's functions are obtained by expressing the generating functional Z as

$$Z[j_\mu\ldots] \equiv \exp(i W[j_\mu\ldots]) \qquad \ldots(3.18)$$

and defining, for example,

$$C(x_1,\mu_1;\ldots;x_n,\mu_n) = i^{n+1} \left.\frac{\delta^n W}{\delta j_{\mu_1}(x_1)\ldots\delta j_{\mu_n}(x_n)}\right|_{j_{\mu_i}=0} \qquad \ldots(3.19)$$

and so on for other fields.

From here we move on to the generator for the one-particle irreducible Green's functions, or vertex functional, which is the Legendre transform of W:

$$\Gamma[A_\mu,\phi,\phi^+,\psi,\bar{\psi}] \equiv W[j^\mu,j,j^+;\eta,\bar{\eta}] - \int(j^\mu A_\mu - j\phi^+ + j^+\phi + \eta\bar{\psi} + \bar{\eta}\psi) \qquad \text{...(3.20)}$$

so that, for instance,

$$\Gamma(x_1,\mu_1;\ldots;x_n,\mu_n) = \frac{\delta^n \Gamma[A_\mu,\ldots]}{\delta A_{\mu_1}(x_1)\ldots\delta A_{\mu_n}(x_n)} \qquad \text{...(3.21)}$$

FIXING A GAUGE:

We now discuss the standard method by which one imposes the gauge condition (3.12) and integrates the gauge degrees of freedom out of the theory. This method, originally expounded by Faddeev and Popov [21], and thus bearing their names, is covered in the references previously cited.

Firstly, we assume that the gauge transformed version of (3.12),

$$F[A_{\mu\Lambda}] = B \qquad \text{...(3.22)}$$

has a unique solution Λ given A_μ, F and B. Therefore, we define

$$\mathcal{F}^{-1}[A_{\mu\Lambda}] \equiv \int[d\Lambda]\,\delta(F[A_{\mu\Lambda}]-B) \qquad \text{...(3.23)}$$

which allows the insertion of a unit operator containing the gauge condition (3.12) into (3.13). So the generating functional Z is now given by

$$Z = \int[dA_\mu d\Lambda\ldots]\,\mathcal{F}[A_{\mu\Lambda}]\,\delta(F[A_{\mu\Lambda}]-B)\exp i\int(\mathcal{L}+\cdots) \qquad \text{...(3.24)}$$

Since the physical S-matrix elements are independent of the choice of gauge, i.e. of B, (3.24) can be integrated with respect to a weight function $\rho(B)$. The infinite gauge volume $\int[d\Lambda]$ is discarded to leave us with

$$Z = \int[dA_\mu\ldots]\,\mathcal{F}[A_{\mu\Lambda}]\,\rho(F[A_{\mu\Lambda}])\exp i\int(\mathcal{L}+\cdots) \qquad \text{...(3.25)}$$

Now, expanding $F[A_{\mu\Lambda}(x)]$ about $\Lambda=0$ gives

$$F[A_\mu(x) - D_\mu\Lambda(x)] = F[A_\mu(x)] - D_\mu\Lambda(x)\frac{\delta F[A_\mu(x)]}{\delta A_\mu(x)} + \cdots \qquad \ldots(3.26)$$

That is,

$$F[A_{\mu\Lambda}(x)] = F[A_\mu(x)] + \int \Lambda(y) M(y,x) d^4y + \cdots \qquad \ldots(3.27)$$

where
$$M(y,x) \equiv \frac{\delta F[A_{\mu\Lambda}(x)]}{\delta \Lambda(y)} \qquad \ldots(3.28)$$

so that

$$\begin{aligned} \mathcal{F}[A_{\mu_\Lambda}(x)] &= \left(\int [d\Lambda]\,\delta(M\Lambda)\right)^{-1} \\ &= \det M \end{aligned} \qquad \ldots(3.29)$$

leading to

$$Z = \int [dA_\mu \ldots]\,\rho(F[A_{\mu\Lambda}])\det M \exp i\int(\mathcal{L} + \cdots) \qquad \ldots(3.30)$$

Thus it is apparent that the outcome of all this is to produce a new term $(\det M)$ in the functional formalism to accompany the gauge fixing term $F[A_\mu]$. The form of the weight function ρ may be taken as an exponential, allowing the addition of a gauge fixing term into the Lagrangian, in either of two ways:

1)
$$\rho(F[A_\mu]) \propto \int [dB]\, e^{-i\int BF[A_\mu]} \qquad \ldots(3.31)$$

where B is a field introduced solely for this purpose, or

2)
$$\rho(F[A_\mu]) \propto e^{i\int a(F[A_\mu])^2} \qquad \ldots(3.32)$$

where a is a constant. This is the Gaussian form, and ρ becomes a δ-function in the limit $a \to \infty$.

There is also a functional representation for $\det M$ as

$$\det M \propto \int [d\chi\, d\bar{\chi}]\, e^{-i\int \bar{\chi} M \chi} \qquad \ldots(3.33)$$

where χ, $\bar{\chi}$ are anticommuting scalar fields, known as the Faddeev-Popov ghosts.

Note that so far in this discussion we have not bothered to distinguish between abelian or non-abelian gauge theory. The technique applies equally well to both (see, e.g., [16]). However, the final outcome hinges on whether M contains a factor coupling the fictitious χ-field to the gauge field.

In the next section the Feynman rules are given for the non-abelian theory with a manifestly covariant gauge choice incorporating the Lorentz gauge of Chapter 2.

3.3 FEYNMAN RULES IN COVARIANT GAUGES

A manifestly covariant gauge condition is

$$\partial^{\mu} A_{\mu} = B \qquad \text{...(3.34)}$$

We call such a gauge with any chosen value of B a *Lorentz* gauge. Equations (3.28) and (3.29) give

$$\mathcal{F}[A_{\mu\lambda}] = \det M \propto \det[\partial . D] \qquad \text{...(3.35)}$$

where we should remember that in full notation D is a matrix in isotopic and space-time indices, that is

$$D_{\mu}^{ab}(x,y) = \delta^{ab} D_{\mu}\, \delta^{4}(x-y)$$

so that $\quad \partial . D = \exp(\mathrm{Tr}\, \ln \partial . D)$

where $\quad \mathrm{Tr}\, K \equiv \sum_{a} \int d^{4}x\, K^{aa}(x)$

The condition (3.34) is introduced by adding a gauge fixing term $\mathcal{L}_{g.f.}$ to the Lagrangian by the method of (3.32),

$$\mathcal{L}_{g.f.} = -\frac{1}{2c}(\partial . A)^{2} \qquad \text{...(3.36)}$$

where c is a constant. Specific values of c give familiar gauges. For instance, $c=0$ is the Landau gauge, $c=1$ is the Fermi gauge, $c=3$ is the Fried-Yennie gauge, and so on [6,15, 20,22,23,24,25].

The corresponding Faddeev-Popov ghost term $\mathcal{L}_{FP}$ due to (3.35) which is added to the Lagrangian by the method of (3.33) is

$$\mathcal{L}_{FP} = -\bar{\chi}\, \partial . D\chi$$
$$= -\bar{\chi}^a . \partial^\mu(\partial_\mu \delta^{ab} - g f^{abc} A_\mu^c)\chi^b \qquad \ldots(3.37)$$

At this stage we can note that it is only because the term '$g\underline{A}\times\ldots$' occurs in the covariant derivative that the ghost field must be included. Otherwise the ghosts decouple from the theory. This is the case in electrodynamics for this and other gauge choices.

The complete generating functional of (3.30) is now

$$Z_{covariant} = \int [dA_\mu d\chi d\bar{\chi}\ldots] \exp i \int (\mathcal{L} - \tfrac{1}{2c}(\partial . A)^2 - \bar{\chi}\partial . D\chi - j^\mu A_\mu + \cdots) \qquad \ldots(3.38)$$

It is now possible to read off the Feynman rules in this gauge. Propagators are obtained by inverting the matrices connecting terms quadratic in A_μ, χ, ϕ and ψ; whilst vertices come from the interaction parts.

It is instructive to go through the process of finding the propagator for the gauge field. To do this we consider the linearised equation of motion:

$$(\partial^2 \eta_{\mu\nu} - \partial_\mu \partial_\nu) A^\nu + \tfrac{1}{c}\partial_\mu \partial_\nu A^\nu = \xi_\mu \qquad \ldots(3.39)$$

Multiplying both sides of (3.39) by ∂^μ gives

$$\partial . A = \frac{c}{\partial^2}\, \partial . \xi \qquad \ldots(3.40)$$

So, combining (3.39) and (3.40), we have

$$A_\mu = \frac{-1}{\partial^2}\left[-\eta_{\mu\nu} + (1-c)\frac{\partial_\mu \partial_\nu}{\partial^2}\right]\xi^\nu$$
$$\equiv \Delta_{\mu\nu}(\partial)\, \xi^\nu \qquad \ldots(3.41)$$

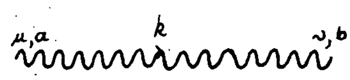
In momentum space, the propagator $\Delta^{ab}_{\mu\nu}(\partial)$ defined in (3.41) reads

$$\Delta^{ab}_{\mu\nu}(k) = \frac{\delta^{ab}}{k}\left[-\eta_{\mu\nu} + (1-c)\frac{k_\mu k_\nu}{k}\right] \qquad \ldots(3.42)$$

The rest of the rules are straightforward enough. For instance, the ghost propagator and vertex are read off immediately as

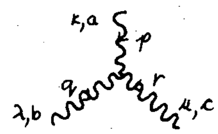
$$\Delta^{ab}_\chi(k) = \frac{\delta^{ab}}{k^2} \qquad \ldots(3.43a)$$

FEYNMAN RULES - COVARIANT GAUGES



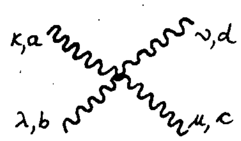
$$\Delta^{ab}_{\mu\nu}(k) = \frac{\delta^{ab}}{k^2}\left[-\eta_{\mu\nu} + (1-c)\frac{k_\mu k_\nu}{k^2}\right]$$

gauge field propagator



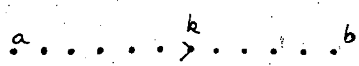
$$\Lambda^{abc}_{\kappa\lambda\mu}(p,q,r) = -igf^{abc}\left[(q-r)_\kappa\eta_{\lambda\mu} + (r-p)_\lambda\eta_{\mu\kappa} + (p-q)_\mu\eta_{\kappa\lambda}\right]$$

gauge field triple vertex



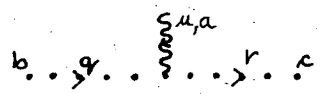
$$\Lambda^{abcd}_{\kappa\lambda\mu\nu} = g^2\left[f^{abe}f^{cde}(\eta_{\kappa\nu}\eta_{\lambda\mu} - \eta_{\kappa\mu}\eta_{\lambda\nu}) + f^{dce}f^{dbe}(\eta_{\kappa\lambda}\eta_{\mu\nu} - \eta_{\kappa\nu}\eta_{\lambda\mu}) + f^{ade}f^{bce}(\eta_{\kappa\mu}\eta_{\lambda\nu} - \eta_{\kappa\lambda}\eta_{\mu\nu})\right]$$

gauge field quadruple vertex



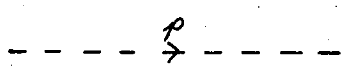
$$\Delta^{ab}_{\chi}(k) = \frac{\delta^{ab}}{k^2}$$

ghost propagator



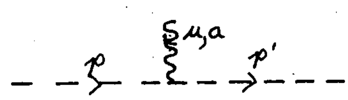
$$\Lambda^{abc}_{\chi\,\mu}(r) = igf^{abc}r_\mu$$

ghost vertex



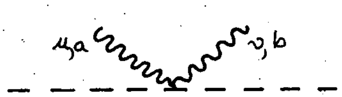
$$\Delta_\phi(p) = \frac{1}{p^2-m^2}$$

scalar meson propagator



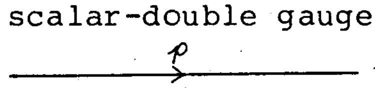
$$\Lambda_{\phi\,\mu}^{\;a}(p',p) = gT^a(p+p')_\mu$$

scalar-single gauge field vertex



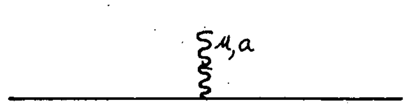
$$\Lambda^{ab}_{\phi\,\mu\nu} = g^2\{T^a,T^b\}\eta_{\mu\nu}$$

scalar-double gauge field vertex



$$\Delta_\psi(p) = \frac{1}{\not p - m}$$

fermion propagator



$$\Lambda_{\psi\,\mu}^{\;a} = gT^a\gamma_\mu$$

fermion vertex

FIGURE 3.1 Feynman rules for the covariant gauge choice $\mathcal{L}_{g.f.} = -\frac{1}{2c}(\partial^\mu A_\mu)^2$.

$$\Gamma^{abc}_{\chi\,\mu}(r) = i g f^{abc} r_\mu \qquad \ldots(3.43b)$$

Notice that in (3.43b), r is the outgoing momentum at the vertex, since in (3.37) the derivative only acts on $\bar{\chi}$.

The Feynman rules for Lorentz covariant gauges are listed in Figure 3.1.

3.4 FEYNMAN RULES IN THE AXIAL GAUGE

The *axial* gauge condition is

$$n^\mu A_\mu = 0 \qquad \ldots(3.44)$$

Equations (3.28) and (3.29) give

$$\mathcal{F}[A_\mu] = \det M \propto \det[n.D] \qquad \ldots(3.45)$$

It is possible to introduce the condition (3.44) via a gauge fixing term along the same lines as (3.36) by defining

$$\mathcal{L}_{gf} \equiv \lim_{\beta\to\infty} \frac{-\beta}{2}(n.A)^2 \qquad \ldots(3.46)$$

The corresponding Faddeev-Popov ghost term similar to (3.37) is

$$\begin{aligned}\mathcal{L}_{FP} &= -\chi\, n.D\chi \\ &= -\bar{\chi}^a n^\mu(\partial_\mu \delta^{ab} - g f^{abc} A^c_\mu)\chi^b \\ &= -\bar{\chi}\; n.\partial[\delta^{ab} - g(n.\partial)^{-1} f^{akc} n.A^c]\chi^b \qquad \ldots(3.47)\end{aligned}$$

It is at this stage that a subtlety occurs in this gauge which is quite fortuitous: the condition $n.A=0$ means that the gauge field term in (3.47) may be dropped and the ghost field decouples [3,10,13,15,20,26].

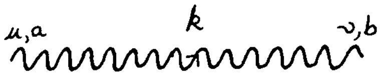
The complete generating functional of (3.30) is thus only

$$Z_{axial} = \lim_{\beta\to\infty} \int [dA_\mu \ldots] \exp i \int \left(\mathcal{L} - \frac{\beta}{2}(n.A)^2 - j^\mu A_\mu + \cdots\right) \qquad \ldots(3.48)$$

Once again we go through the process of finding the vector meson propagator. The linearised equation of motion is now

$$(\partial^2 \eta_{\mu\nu} - \partial_\mu \partial_\nu)A^\nu + \beta n_\mu n_\nu A^\nu = \xi_\mu \qquad \ldots(3.49)$$

FEYNMAN RULES - AXIAL GAUGE

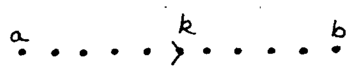


gauge field propagator

$$\Delta^{ab}_{\mu\nu}(k) = \frac{\delta^{ab}}{k^2}\left[-\eta_{\mu\nu} + \frac{n_\mu k_\nu + k_\mu n_\nu}{(n.k)} - \frac{n^2 k_\mu k_\nu}{(n.k)^2}\right]$$

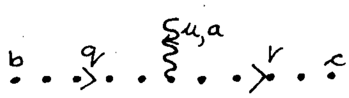
$$\longrightarrow \Delta^{ab}_{\mu\nu}(k) = \frac{\delta^{ab}}{k^2}\left[-\eta_{\mu\nu} + \frac{\eta_{\mu 0} k_\nu + k_\mu \eta_{\nu 0}}{k_0} - \frac{k_\mu k_\nu}{k_0^2}\right]$$

when $\quad n_\mu = (1, 0, 0, 0)$



ghost propagator

$\Delta^{ab}_{\chi}(k)$ is arbitrary



ghost vertex

$$\Lambda^{abc}_{\chi\,\mu}(r) = 0$$

(All other rules as for Fig. 3.1)

FIGURE 3.2 Feynman rules for the axial gauge choice $\mathcal{L}_{g.f.} = \lim_{\beta\to\infty} -\frac{\beta}{2}(n^\mu A_\mu)^2$. Since the ghost field does not couple to the gauge field in this case, the ghost propagator may be taken as arbitrary, although equation (3.47) gives the formal result $\Delta^{ab}_{\chi}(k) = \frac{\delta^{ab}}{n.k}$.

Multiplying (3.49) on both sides by n^μ gives

$$\partial.A = (\partial.n)^{-1}\,[\,\partial^2(n.A) + \beta n^2(n.A) - n.\xi\,] \qquad \ldots(3.50a)$$

and on both sides by ∂^μ gives

$$n.A = [\,\beta(\partial.n)]^{-1}\,\partial.\xi \qquad \ldots(3.50b)$$

So, combining (3.49) and (3.50), we have

$$A_\mu = \frac{1}{\partial^2}\Big[\eta_{\mu\nu} - \frac{\partial_\mu n_\nu + n_\mu \partial_\nu}{(\partial.n)} + \frac{n^2\partial_\mu\partial_\nu}{(\partial.n)^2} + \frac{\partial^2\partial_\mu\partial_\nu}{\beta(\partial.n)^2}\Big]\xi^\nu$$
$$\equiv \Delta_{\mu\nu}(\partial)\,\xi^\nu \qquad \ldots(3.51)$$

Taking the limit $\beta\to\infty$ gives the axial gauge propagator defined by (3.51) in momentum space as

$$\Delta^{ab}_{\mu\nu}(k) = \frac{\delta^{ab}}{k^2}\Big[-\eta_{\mu\nu} + \frac{n_\mu k_\nu + k_\mu n_\nu}{(k.n)} - \frac{n^2 k_\mu k_\nu}{(k.n)^2}\Big] \qquad \ldots(3.52)$$

The absence of ghosts leads to particularly simple Feynman rules, which are listed in Figure 3.2.

3.5 FEYNMAN RULES IN THE COULOMB GAUGE

The Coulomb gauge condition is

$$\underline{\partial}.\underline{A} = 0 \qquad \ldots(3.53)$$

It is convenient to generalise (3.53) along the lines of equations (2.86) and (2.87), so that the condition becomes [23,27]

$$\Big(\partial^\mu - n^\mu \frac{\partial.n}{n^2}\Big)A_\mu = 0 \qquad \ldots(3.54)$$

with (3.53) being recovered when n^μ is taken to be unit time-like.

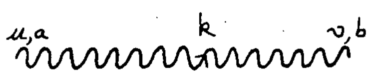
With condition (3.54), equations (3.28) and (3.29) give

$$\mathcal{F}[A_\mu] = \det M \propto \det\Big(\partial.D - n.D\,\frac{\partial.n}{n^2}\Big) \qquad \ldots(3.55)$$

We introduce (3.54) into the functional formalism by adding the gauge fixing term

$$\mathcal{L}_{gf} \equiv \lim_{\alpha\to\infty} -\frac{\alpha}{2}\Big(\partial.A - n.A\,\frac{\partial.n}{n^2}\Big)^2 \qquad \ldots(3.56)$$

FEYNMAN RULES - COULOMB GAUGE

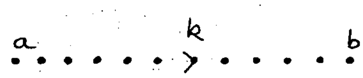


$$\Delta^{ab}_{\mu\nu}(k) = \frac{\delta^{ab}}{k^2}\left[-\eta_{\mu\nu} + \frac{(k.n)(k_\mu n_\nu + n_\mu k_\nu)}{(k.n)^2 - k^2 n^2} - \frac{n^2 k_\mu k_\nu}{(k.n)^2 - k^2 n^2}\right]$$

gauge field propagator

$$\rightarrow \quad \Delta^{ab}_{\mu\nu}(k) = \frac{\delta^{ab}}{k^2}\left[-\eta_{\mu\nu} + \frac{k_0(\eta_{\mu 0} k_\nu + k_\mu \eta_{\nu 0})}{\underline{k}^2} - \frac{k_\mu k_\nu}{\underline{k}^2}\right]$$

when $\quad n_\mu = (1,0,0,0)$

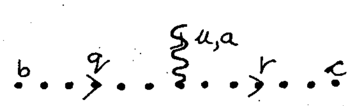


$$\Delta^{ab}_{\chi}(k) = \frac{-n^2\,\delta^{ab}}{(k.n)^2 - k^2 n^2}$$

ghost propagator

$$\rightarrow \quad \Delta^{ab}_{\chi}(k) = -\frac{\delta^{ab}}{\underline{k}^2}$$

when $\quad n_\mu = (1,0,0,0)$



$$\Lambda^{abc}_{\chi\,\mu}(r) = i g f^{abc}\left[r_\mu - \frac{r.n}{n^2} n_\mu\right]$$

ghost vertex

$$\rightarrow \quad \Lambda^{abc}_{\chi\,\mu}(r) = i g f^{abc}\left[r_\mu - \eta_{\mu 0} r_0\right]$$

when $\quad n_\mu = (1,0,0,0)$

(All other rules as for Fig. 3.1).

FIGURE 3.3 Feynman rules for the Coulomb gauge choice $\mathcal{L}_{g.f.} = \lim_{\alpha\to\infty} -\frac{\alpha}{2}\left(\partial^\mu A_\mu - n^\mu A_\mu \frac{\partial^\nu n_\nu}{n^2}\right)^2$.

and the corresponding Faddeev-Popov ghost term is

$$\mathcal{L}_{FP} = -\bar{\chi}\left(\partial.D - n.D\,\frac{\partial.n}{n^2}\right)\chi \qquad \ldots(3.57)$$

Thus, the complete generating functional is

$$Z_{Coulomb} = \lim_{\alpha\to\infty} \int [dA_\mu d\chi d\bar{\chi}]\times$$
$$\times \exp i\int\left(\mathcal{L} - \frac{\alpha}{2}\left(\partial.A - n.A\,\frac{\partial.n}{n^2}\right)^2 - \bar{\chi}\left(\partial.D - n.D\frac{\partial.n}{n^2}\right)\chi + \ldots\right) \qquad \ldots(3.58)$$

To find the vector meson propagator, we use the linearised equation of motion

$$(\partial^2\eta_{\mu\nu} - \partial_\mu\partial_\nu)A^\nu + \alpha\left(\partial_\mu - \frac{\partial.n}{n^2}n_\mu\right)\left(\partial_\nu - \frac{\partial.n}{n^2}n_\nu\right)A^\nu = \xi_\mu$$
$$\ldots(3.59)$$

Multiplying (3.59) on both sides by ∂^μ gives

$$\partial.A - \frac{\partial.n}{n^2}\,n.A = \partial.\xi/\alpha\left[\partial^2 - \frac{(\partial.n)^2}{n^2}\right] \qquad \ldots(3.60a)$$

and on both sides by n^μ gives

$$n.A = \frac{1}{\partial^2}\left[n.\xi + (\partial.n)(\partial.A)\right] \qquad \ldots(3.60b)$$

So, combining (3.59) and (3.60), we have

$$A_\mu = \frac{1}{\partial^2}\left[\eta_{\mu\nu} + \frac{\frac{\partial.n}{n^2}(\partial_\mu n_\nu + n_\mu\partial_\nu) - \partial_\mu\partial_\nu}{\partial^2 - \frac{(\partial.n)^2}{n^2}} + \frac{\partial^2\,\partial_\mu\partial_\nu}{\alpha\left[\partial^2 - \frac{(\partial.n)^2}{n^2}\right]^2}\right]\xi^\nu$$

$$\equiv \Delta_{\mu\nu}(\partial)\,\xi^\nu \qquad \ldots(3.61)$$

Taking the limit $\alpha\to\infty$ gives the Coulomb gauge propagator defined by (3.61) in momentum space as

$$\Delta^{ab}_{\mu\nu}(k) = \frac{\delta^{ab}}{k^2}\left[-\eta_{\mu\nu} + \frac{(k.n)}{(k.n)^2 - k^2n^2}(k_\mu n_\nu + n_\mu k_\nu) - \frac{n^2 k_\mu k_\nu}{(k.n)^2 - k^2 n^2}\right]$$
$$\ldots(3.62)$$

Other Feynman rules are straightforward, and they are listed in Figure 3.3.

3.6 FEYNMAN RULES IN A GENERAL GAUGE

Following Frenkel and Taylor [27], it is apparent that the gauge fixing terms (3.46) and (3.56), for the axial and Coulomb gauges respectively, may be quite simply combined into the general gauge fixing term [28]

$$\mathcal{L}_{gf} = -\frac{1}{2}\left[\beta n^\mu A_\mu \frac{\partial.n}{n^2} + \alpha\left(\partial^\mu - n^\mu\frac{\partial.n}{n^2}\right)A_\mu\right]^2$$
$$\to -\frac{1}{2}\left[(b^2 - a^2)n^\mu A_\mu \frac{\partial.n}{n^2} + a^2\,\partial^\mu A_\mu\right]^2 \qquad \ldots(3.63)$$

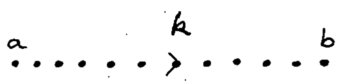
FEYNMAN RULES - GENERAL GAUGE

gauge field propagator

$$\Delta^{ab}_{\mu\nu}(k) = \frac{\delta^{ab}}{k^2}\left\{-\eta_{\mu\nu} + \frac{(b^2-a^2)(k.n)}{(b^2-a^2)(k.n)^2+a^2k^2n^2}(k_\mu n_\nu + n_\mu k_\nu) - \frac{(1-a^4)k^2n^4+(b^2-a^2)^2(k.n)^2n^2}{[(b^2-a^2)(k.n)^2+a^2k^2n^2]^2}\,k_\mu k_\nu\right\}$$

$$\longrightarrow \Delta^{ab}_{\mu\nu}(k) = \frac{\delta^{ab}}{k^2}\left\{-\eta_{\mu\nu} + \frac{(b^2-a^2)k_0}{(b^2k_0^2-a^2\underline{k}^2)}(k_\mu \eta_{\nu 0} + \eta_{\mu 0}k_\nu) - \frac{(1-a^4)\underline{k}^2+(b^2-a^2)k_0^2}{(b^2k_0^2-a^2\underline{k}^2)^2}\,k_\mu k_\nu\right\}$$

when $n_\mu = (1,0,0,0)$

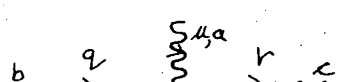


ghost propagator

$$\Delta^{ab}_{\chi}(k) = \frac{\delta^{ab}a^2n^2}{a^2k^2n^2+(b^2-a^2)(k.n)^2}$$

$$\longrightarrow \Delta^{ab}_{\chi}(k) = \frac{\delta^{ab}a^2}{b^2k_0^2-a^2\underline{k}^2}$$

when $n_\mu = (1,0,0,0)$



ghost vertex

$$\Lambda^{abc}_{\chi\,\mu}(r) = igf^{abc}\left[r_\mu + \frac{b^2-a^2}{a^2}\frac{r.n}{n^2}n_\mu\right]$$

$$\longrightarrow \Lambda^{abc}_{\chi\,\mu}(r) = igf^{abc}\left[r_\mu + \frac{b^2-a^2}{a^2}\eta_{\mu 0}r_0\right]$$

when $n_\mu = (1,0,0,0)$

(All other rules as for Fig. 3.1)

FIGURE 3.4 Feynman rules for the general gauge choice $\mathcal{L}_{g.f.} = -\frac{1}{2}\left[(b^2-a^2)n^\mu A_\mu \frac{\partial^\nu n_\nu}{n^2} + a^2\partial^\mu A_\mu\right]^2$.

where for the sake of simplicity in future working we have made the substitutions $\alpha \equiv a^2$ and $\beta \equiv b^2$.

(3.63) is interesting because it reduces in various limits to the three gauge choices already discussed. These are

$a^2 = 0$: Axial Gauges.

$b^2 \to \infty$ yields the usual axial gauge

$b^2 = 0$: Coulomb Gauges

$a^2 \to \infty$ yields the usual Coulomb gauge

$a^2 = b^2 \equiv \alpha^{-1/4}$: Covariant gauges

We consider (3.63) as corresponding to a general gauge condition

$$(b^2 - a^2) \frac{\partial . n}{n^2} n^\mu A_\mu + a^2 \partial^\mu A_\mu = 0 \qquad \text{...(3.64)}$$

Then (3.28) and (3.29) give

$$\mathcal{F}[A_\mu] = \det M \propto \det \{ [(a^2 - b^2) \frac{\partial . n}{n^2} n^\mu - a^2 \partial^\mu] . D_\mu \} \qquad \text{...(3.65)}$$

So the Faddeev-Popov ghost term is

$$\mathcal{L}_{FP} = -\bar{\chi} \left[\frac{a^2 - b^2}{a^2} \frac{\partial . n}{n^2} n . D - \partial . D \right] \chi \qquad \text{...(3.66)}$$

yielding altogether the complete generating functional

$$Z_{general} = \int [dA_\mu d\chi d\bar{\chi}] \exp i \int (\mathcal{L} - \tfrac{1}{\alpha} [(b^2 - a^2) \tfrac{\partial . n}{n^2} n . A + a^2 \partial . A]^2 - \bar{\chi} [\tfrac{a^2 - b^2}{a^2} \tfrac{\partial . n}{n^2} n . D - \partial . D] \chi + \cdots) \qquad \text{...(3.67)}$$

The general gauge vector meson propagator is derived in Appendix C and can be found in equation (C12) as

$$\Delta^{ab}_{\mu\nu}(k) = \frac{\delta^{ab}}{k^2} \left\{ -\eta_{\mu\nu} + \frac{(b^2 - a^2) k . n}{(b^2 - a^2)(k . n)^2 + a^2 k^2 n^2} (k_\mu n_\nu + n_\mu k_\nu) - \frac{(1 - a^4) k^2 n^4 + (b^2 - a^2)^2 (k . n)^2 n^2}{[(b^2 - a^2)(k . n)^2 + a^2 k^2 n^2]^2} k_\mu k_\nu \right\} \qquad \text{...(3.68)}$$

The Feynman rules in this general gauge are listed in Figure 3.4.

3.7 RENORMALIZATION CONSTANTS

In this section we list the basic infinities of non-abelian gauge theory as described by the Lagrangian (3.7), taking into account the possible presence of fictitious scalar ghost particles through the Faddeev-Popov term (3.33). We use a notation dictated in part by tradition and in part by economy. Wave-function renormalization constants are labelled Z_3 and charge or vertex renormalization constants by Z_1; others are in an obvious notation.

The purpose of renormalization is to remove infinities associated with loop calculations so as to ensure that the complete propagators and proper scattering vertices are finite to every order in the gauge coupling constant g. Introductions to the technique can be found in references [3,7, 10,13,14,15,17,18,19]. The primitive divergences of the theory are given in Figure 3.5. Later in Section 3.8 when deriving identities between the renormalization constants we will introduce more fields and divergences, but for the moment those in Fig. 3.5 will suffice.

The complete propagators and proper scattering vertices are listed in Table 3.6. From calculations of the one loop self-energies ($\Pi, \Sigma, \ldots$) and vertex corrections (Γ), it is then possible to determine the renormalization constants to that order [3,29].

The renormalised fields, masses and coupling constant are obtained from the bare ones by rescaling [3,18]:

$$\left.\begin{aligned} A_0 &= Z_{3A}^{1/2} A \quad, & \chi_0 &= Z_{3\chi}^{1/2} \chi \\ \phi_0 &= Z_{3\phi}^{1/2} \phi \quad, & \psi_0 &= Z_{3\psi}^{1/2} \psi \\ M_0^2 &= Z_M M^2 \quad, & m_0 &= Z_m m \\ g_0 &= Z_{1A} Z_{3A}^{-3/2} g & & \end{aligned}\right\} \qquad \ldots (3.69)$$

PRIMITIVE DIVERGENCES

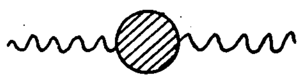
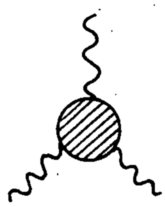
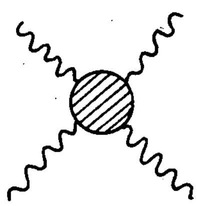
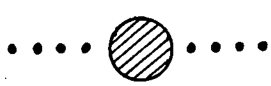
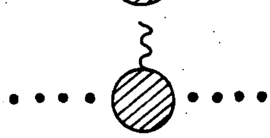
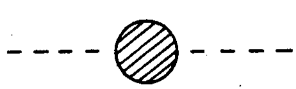
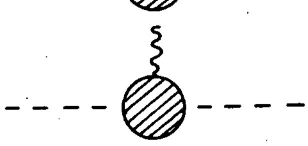
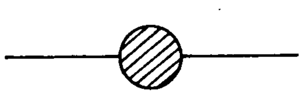
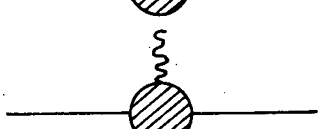
Correction, and associated renormalization constant		Description
	Z_{3A}	vector meson wave-function
	Z_{1A}	vector meson triple vertex correction
	Z_{4A}	vector meson quadruple vertex correction
	$Z_{3\chi}$	ghost wave-function
	$Z_{1\chi}$	ghost vertex
	$Z_{3\phi}$	scalar meson wave-function
	$Z_{1\phi}$	scalar vertex correction
	$Z_{3\psi}$	fermion wave-function
	$Z_{1\psi}$	fermion vertex correction

FIGURE 3.5 The basic infinities and associated renormalization constants for non-abelian gauge theory.

COMPLETE PROPAGATORS AND PROPER SCATTERING VERTICES

Field	Propagators and Vertices
vector meson	$[\Delta'^{ab}_{\mu\nu}(k)]^{-1} = Z_{3A}\delta^{ab}(-\eta_{\mu\nu}k^2 + k_\mu k_\nu) + \Pi^{ab}_{\mu\nu}(k)$
ghost	$[\Delta'^{ab}_{\chi}(k)]^{-1} = Z_{3\chi}\delta^{ab}k^2 + \Pi^{ab}(k)$
	$g^{-1}\Lambda'^{abc}_{\chi\mu}(r) = Z_{1\chi} i f^{abc} r_\mu + \Gamma^{abc}_{\mu}(r)$
scalar meson	$[\Delta'_{\phi}(p)]^{-1} = Z_{3\phi}(p^2 - m_0^2) + \Pi(p)$
	$g^{-1}\Lambda'_{\phi\mu}(p',p) = Z_{1\phi}T^a(p'+p)_\mu + \Gamma^a_\mu(p';p)$
fermion	$[\Delta'_{\psi}(p)]^{-1} = Z_{3\psi}(\not{p} - m_0) + \Sigma(p)$
	$g^{-1}\Lambda'^{a}_{\psi\mu}(p';p) = Z_{1\psi}T^a\gamma_\mu + \Gamma^a_\mu(p';p)$

TABLE 3.6 Complete propagators and proper scattering vertices for non-abelian gauge theory.

3.8 GAUGE INVARIANCE IDENTITIES

In this section we progress from the Ward-Takahashi identities [30,31] in quantum electrodynamics through to their generalizations as the Slavnov-Taylor [32,33] and Becchi-Rouet-Stora [34] identities in non-abelian gauge theory. Since our purpose is to be illustrative rather than exhaustive, we will use the covariant gauges of Section 3.3 throughout.

WARD-TAKAHASHI IDENTITIES

The generating functional in the covariant gauges in quantum electrodynamics is

$$Z[j_\mu,\eta,\bar\eta] = \int [dA_\mu d\psi d\bar\psi] \exp i\int \left(\mathcal{L} - \tfrac{1}{2\alpha}(\partial.A)^2 - j^\mu A_\mu - \bar\eta\psi - \bar\psi\eta\right) \qquad \ldots(3.70)$$

The measure $[dA_\mu d\psi d\bar\psi]$ and invariant Lagrangian $\mathcal{L}$, where

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar\psi\gamma^\mu(i\partial_\mu + eA_\mu - m)\psi \qquad \ldots(3.71)$$

remain the same under the following infinitessimal gauge transformations:

$$\left.\begin{aligned} \Delta A_\mu(x) &= -\partial_\mu \Lambda(x) \\ \Delta\psi(x) &= ie\Lambda(x)\psi(x) \\ \Delta\bar\psi(x) &= -ie\bar\psi(x)\Lambda(x) \end{aligned}\right\} \qquad \ldots(3.72)$$

However, the source and gauge-fixing terms <u>do</u> change by

$$\Delta\left[-\tfrac{1}{2\alpha}(\partial.A)^2 - j.A - \bar\psi\eta - \bar\eta\psi\right] = \tfrac{1}{\alpha}(\partial.A)(\partial\Lambda) + j.\partial\Lambda - ie\bar\eta\Lambda\psi + ie\bar\psi\Lambda\eta \qquad \ldots(3.73)$$

So we have

$$\begin{aligned} 0 &= \int [dA_\mu d\psi d\bar\psi]\left[\tfrac{1}{\alpha}\partial.\partial^2 A + \partial.j - ie\bar\eta\psi + ie\bar\psi\eta\right]\exp i\int(\mathcal{L}+\cdots) \\ &= \left[\tfrac{1}{\alpha}\partial^2\partial_\mu i\frac{\delta}{\delta j_\mu} - \partial^\mu j_\mu + e\bar\eta\frac{\delta}{\delta\bar\eta} - e\eta\frac{\delta}{\delta\eta}\right] Z[j_\mu,\eta,\bar\eta] \end{aligned} \qquad \ldots(3.74)$$

Equation (3.74) is the generalised Ward-Takahaski, or fundamental gauge, identity for quantum electrodynamics [3,9,16, 18,19,22,23,24].

Switching to connected Green's functions through (3.18) we get

$$\partial^\mu j_\mu = \left[\frac{i}{c} \partial^2 \partial_\mu \frac{\delta}{\delta j_\mu} + e\bar{\eta} \frac{\delta}{\delta \bar{\eta}} - e\eta \frac{\delta}{\delta \eta} \right] W[j_\mu, \eta, \bar{\eta}] \qquad \text{...(3.75)}$$

and thence to the effective action via (3.20):

$$-\frac{i}{c} \partial^2 \partial.A(x) - \partial_\mu \frac{\delta \Gamma}{\delta A_\mu(x)} - e \frac{\delta \Gamma}{\delta \psi} \psi + e \bar{\psi} \frac{\delta \Gamma}{\delta \bar{\psi}} = 0 \qquad \text{...(3.76)}$$

The usual simple Ward identities come from taking field functional derivatives, at $A_\mu = \psi = \bar{\psi} = 0$, of (3.76). For instance, taking $\delta / \delta A_\nu(y)$ of (3.76) yields

$$-\frac{i}{c} \partial^2 \partial_\mu \eta^{\mu\nu} \delta^4(x-y) - \partial^\mu \frac{\delta^2 \Gamma}{\delta A_\mu(x) \delta A_\nu(y)}$$

$$= -\frac{i}{c} \partial^2 \partial_\mu \eta^{\mu\nu} \delta^4(x-y) - \partial^\mu [\Delta^{-1}_{\mu\nu}(x,y) + \Pi_{\mu\nu}(x,y)] = 0 \qquad \text{...(3.77)}$$

where $\Pi_{\mu\nu}(x,y)$ is the self-energy correction to the bare inverse propagator $\Delta^{-1}_{\mu\nu}(x,y)$. Transferring to momentum space, (3.77) becomes

$$\frac{1}{c} k^2 k_\nu + k^\mu [-\eta_{\mu\nu} k^2 + (1 - \tfrac{1}{c}) k_\mu k_\nu + \Pi_{\mu\nu}(k)] = k^\mu \Pi_{\mu\nu}(k) = 0 \qquad \text{...(3.78)}$$

That is, the transversality of vacuum polarization follows from the Ward-Takahashi identity (3.74).

The above procedure can be followed through for the axial and Coulomb gauges as well.

SLAVNOV-TAYLOR IDENTITIES

The generating functional in the covariant gauges in non-abelian gauge theory

$$Z[j_\mu, j, j^+, \eta, \bar{\eta}] = \int [dA_\mu d\chi d\bar{\chi} \ldots] \exp i \int (\mathcal{L} - \tfrac{1}{2c}(\partial.A)^2 - \bar{\chi} \partial.D\chi - j^\mu A_\mu + \cdots) \qquad \text{...(3.79)}$$

Neglecting scalar meson and fermion fields for simplicity, it is apparent that the measure involves a functional determinant and this makes things more complicated than in the abelian case. Now, transformations are required which leave $\int[dA_\mu\ldots]\det(\partial.D)$ invariant. For the gauge field it is [32,35]

$$A_\mu \to A_{\mu\Lambda} \equiv A_\mu + D_\mu(\partial.D)^{-1}\Lambda \qquad \ldots(3.80)$$

The $(\partial.D)^{-1}$ term acts like a Green's function for a scalar particle in an external potential A_μ, and means that (3.80) is a non-local transformation. Despite this complication it is still possible to obtain useful results. Under (3.80) the source and gauge-fixing terms change by

$$\Delta\left[\frac{1}{2c}(\partial.A)^2 - j^\mu A_\mu\right] = \frac{1}{c}(\partial.A)\Lambda - j.D(\partial.D)^{-1}\Lambda \qquad \ldots(3.81)$$

So we arrive at

$$\left[\frac{i\partial_\mu}{c}\cdot\frac{\delta}{\delta j_\mu(x)} - \int j(y)\,D[i\tfrac{\delta}{\delta j}](\partial.D[i\tfrac{\delta}{\delta j}])^{-1}(y,x)\right]Z = 0 \qquad \ldots(3.82)$$

which is the Slavnov-Taylor identity [7,9,12,18,22].

Applying $\delta/\delta j_\nu(y)$ to (3.82) gives

$$\frac{i}{c}\partial_{\mu x}\frac{\delta^2 Z}{\delta j_\mu(x)\,\delta j_\nu(y)} - \int D^\nu[i\tfrac{\delta}{\delta j_\nu(y)}](\partial.D_\mu[i\tfrac{\delta}{\delta j_\mu(x)}])^{-1}(y,x)\,Z = 0$$

$$\Rightarrow \quad \frac{i}{c}\partial_{\mu x}\partial_{\nu y} - \delta^4(x-y)Z = 0 \qquad \ldots(3.83)$$

Changing to the connected vacuum functional W gives (3.83) as

$$\frac{i}{c}\partial_{\mu x}\partial_{\nu y}\left.\frac{\delta^2 W}{\delta j_\mu(x)\,\delta j_\nu(y)}\right|_{j=0} = \delta^4(x-y) \qquad \ldots(3.84)$$

which has the momentum space form

$$\frac{1}{c}k^\mu k^\nu \Delta_{\mu\nu}(k) = 1 \qquad \ldots(3.85)$$

Since the bare propagator already satisfies (3.85), the self-energy correction is again found to be transverse as in the abelian case, i.e.

$$k^\mu k^\nu \Pi_{\mu\nu}(k) = 0 \qquad \ldots(3.86)$$

In the Coulomb gauge, the procedure is much the same, with non-local $(\underline{\partial}.\underline{D})^{-1}$ terms occurring in the identities.

However, in the axial gauge the identities become particularly simple, and follow the 'naive' form of the Ward-Takahashi identities in quantum electrodynamics [18,36,37,38,39].

BECCHI-ROUET-STORA IDENTITIES

It is possible to find variations amongst A_μ and χ which leave the Lagrangian, gauge fixing term and Faddeev-Popov ghost term, and the measure, invariant. The trick is to write the gauge function Λ as [34]

$$\Lambda^a = \chi^a \omega \qquad \ldots(3.87)$$

where Λ^a is a Bose quantity, χ^a is a Fermi quantity, and ω is some anticommuting global quantity, so that $\omega^2=0$. Under a change of the gauge field

$$\Delta A_\mu^a = D_\mu \Lambda^a = D_\mu \chi^a \omega \qquad \ldots(3.88)$$

the change in $\mathcal{L} + \mathcal{L}_{g.f.} + \mathcal{L}_{FP}$ is

$$\begin{aligned}\Delta(\mathcal{L} + \mathcal{L}_{g.f.} + \mathcal{L}_{FP}) &= \Delta\left[-\tfrac{1}{2\alpha}(\partial.A^a)^2 + \bar{\chi}^a \partial.D^{ab}\chi^b\right] \\ &= -\tfrac{1}{\alpha}(\partial.A^a)(\partial.D\chi^a\omega) + \Delta\bar{\chi}^a \partial.D^{ab}\chi^b \\ &\quad + \bar{\chi}^a \partial.\Delta D^{ab}\chi^b + \bar{\chi}^a\partial.D^{ab}\Delta\chi^b \end{aligned} \qquad \ldots(3.89)$$

and total cancellation is achieved in (3.89) if, in addition to (3.88), we have

$$\left.\begin{aligned}\Delta\chi^a &= -\tfrac{1}{2} g f^{abc}\chi^b\chi^c\omega \\ \Delta\bar{\chi}^a &= -\tfrac{1}{\alpha}(\partial.A^a)\omega\end{aligned}\right\} \qquad \ldots(3.90a)$$

Besides (3.90a) there is also

$$\left.\begin{aligned}\Delta(D_\mu \underline{\chi}) &= 0 \\ \Delta(\underline{\chi}\times\underline{\chi}) &= 0\end{aligned}\right\} \qquad \ldots(3.90b)$$

and so not only is (3.89) invariant under (3.88) and (3.90a) but also

$$\Delta^2 \underline{A}_\mu = \Delta^2 \underline{\chi} = 0 \qquad \ldots(3.91)$$

and the measure $[dA_\mu d\chi d\bar{\chi}]$ is also invariant.

Because the ghost field is so intimately tied in with the Becchi-Rouet-Stora (BRS) transformations, we add source terms

$$\mathcal{L}_{\text{ghost sources}} = -\bar{J}\chi - \bar{\chi}J - I^{\mu}D_{\mu}\chi - \tfrac{1}{2}g\,I.\underline{\chi}\times\underline{\chi} \qquad \ldots(3.92)$$

to the functional formalism so that identities may be derived. The change in all the source fields due to the BRS transformation (3.89), (3.90a) is then

$$\Delta(\text{sources}) = j^{\mu}.D_{\mu}\underline{\chi}\,\omega - \tfrac{1}{2}g\,\underline{J}.\underline{\chi}\times\underline{\chi}\,\omega - \tfrac{1}{c}(\partial.\underline{A})\,\omega.\underline{\bar{J}} \qquad \ldots(3.93)$$

And so we arrive at

$$\left[j_{\mu}.\frac{\delta}{\delta \underline{I}_{\mu}} + \underline{\bar{J}}.\frac{\delta}{\delta \underline{I}} + \frac{1}{c}\,\underline{J}.\partial_{\mu}\frac{\delta}{\delta j_{\mu}}\right]Z = 0 \qquad \ldots(3.94)$$

which is the BRS identity [12,17,18,19,22].

For the connected Green's functions (3.94) becomes

$$\left[j_{\mu}.\frac{\delta}{\delta \underline{I}_{\mu}} + \underline{\bar{J}}.\frac{\delta}{\delta \underline{I}} + \frac{1}{c}\,\underline{J}.\partial\frac{\delta}{\delta j_{\mu}}\right]W = 0 \qquad \ldots(3.95)$$

and thus the identity for the effective action is

$$\frac{\delta\Gamma}{\delta \underline{A}_{\mu}}.\frac{\delta\Gamma}{\delta \underline{I}^{\mu}} - \frac{\delta\Gamma}{\delta \underline{\chi}}.\frac{\delta\Gamma}{\delta \underline{I}} + \frac{1}{c}\,\partial.\underline{A}\,\frac{\delta\Gamma}{\delta \underline{\bar{\chi}}} = 0 \qquad \ldots(3.96)$$

Varying the full Lagrangian with respect to $\bar{\chi}$ gives the equation of motion

$$\partial.D\,\underline{\chi} = \underline{J} \quad \text{or} \quad \partial^{\mu}\frac{\delta\Gamma}{\delta \underline{I}^{\mu}} = \frac{\delta\Gamma}{\delta \underline{\bar{\chi}}} \qquad \ldots(3.97)$$

If ghost number 1 is affixed to χ and 0 to $\mathcal{L}$, then there are the assignments

$$\left.\begin{aligned} N(\chi) &= N(J) = 1 \\ N(\bar{\chi}) &= N(\omega) = N(\bar{J}) = N(I_{\mu}) = -1 \\ N(A_{\mu}) &= N(j_{\mu}) = 0 \\ N(I) &= -2 \end{aligned}\right\} \qquad \ldots(3.98)$$

Since physically interesting Green's functions are those which involve processes having zero ghost number, the only meaningful identities are those obtained by differentiating (3.96) with respect to the fields so as to leave $N=0$. For instance, from (3.97) we obtain

$$\partial^{\mu} \frac{\delta^2 \Gamma}{\delta I^{\mu} \delta \chi} = \frac{\delta^2 \Gamma}{\delta \chi \delta \bar{\chi}} \qquad \ldots(3.99)$$

and

$$\partial^{\mu} \frac{\delta^3 \Gamma}{\delta A_{\nu} \delta I^{\mu} \delta \chi} = \frac{\delta^3 \Gamma}{\delta A_{\nu} \delta \chi \delta \bar{\chi}} \qquad \ldots(3.100)$$

Taking $\delta^3/\delta A_{\nu} \delta A_{\lambda} \delta \chi$ of (3.96) gives

$$\frac{\delta^3 \Gamma}{\delta A_{\lambda} \delta A_{\nu} \delta A_{\mu}} \cdot \frac{\delta^2 \Gamma}{\delta I^{\mu} \delta \chi} + \frac{\delta^2 \Gamma}{\delta A_{\nu} \delta A_{\mu}} \cdot \frac{\delta^3 \Gamma}{\delta A_{\lambda} \delta I^{\mu} \delta \chi} = \frac{1}{\alpha} \partial^{\mu} \frac{\delta^3 \Gamma}{\delta A_{\lambda} \delta \chi \delta \bar{\chi}} \qquad \ldots(3.101)$$

Again, taking $\delta^2/\delta \chi \delta \bar{\chi}$ of (3.96) gives

$$\partial_{\mu} \frac{\delta^3 \Gamma}{\delta A_{\mu} \delta \chi \delta \bar{\chi}} = \frac{\delta^3 \Gamma}{\delta I \delta \chi \delta \bar{\chi}} \qquad \ldots(3.102)$$

and taking $\delta^3/\delta A_{\nu} \delta A_{\lambda} \delta A_{\kappa}$ of (3.96) gives

$$\frac{\delta^4 \Gamma}{\delta A_{\mu} \delta A_{\nu} \delta A_{\lambda} \delta A_{\kappa}} = 0 \qquad \ldots(3.103)$$

Besides these identities, one can show the transversality of the gauge field self-energy by taking $\delta^2/\delta A_{\nu} \delta \chi$ of (3.96), which gives

$$\int d^4x \, \frac{\delta^2 \Gamma}{\delta A_{\nu}(y) \delta I_{\mu}(x)} \cdot \frac{\delta^2 \Gamma}{\delta A_{\mu}(x) \delta \chi(z)} = \frac{1}{\alpha} \int d^4x \, \delta^{\nu}(y-x) \frac{\delta^2 \Gamma}{\delta \bar{\chi}(x) \delta \chi(z)} \qquad \ldots(3.104)$$

which is, in momentum space,

$$\left[k^2 \eta_{\mu\nu} - k_{\mu} k_{\nu} (1 - \tfrac{1}{\alpha}) \Pi_{\mu\nu}(k)\right] \frac{\delta^2 \Gamma}{\delta I_{\mu} \delta \chi} = \frac{1}{\alpha} k_{\nu} \frac{\delta^2 \Gamma}{\delta \chi \delta \bar{\chi}} \qquad \ldots(3.105)$$

whence, using (3.99),

$$k^{\mu} \Pi_{\mu\nu}(k) = 0 \qquad \ldots(3.106)$$

The identities (3.99), (3.100), (3.101), (3.102), (3.103) may be translated into relations between renormalization constants by introducing extra primitive divergences due to the fields I^{μ} and I. These are listed in Figure 3.7, and supplement those given in Figure 3.5.

The full renormalized Lagrangian with gauge fixing, ghost and BRS terms is

EXTRA PRIMITIVE DIVERGENCES

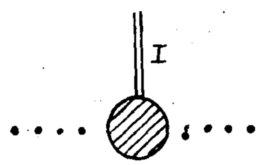
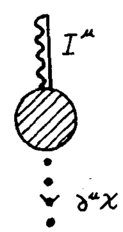
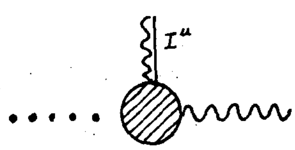
Correction, and associated renormalization constant		Description
	Z_{1I}	I -ghost vertex
	Z_{3I_μ}	I^μ -ghost vertex
	Z_{1I_μ}	I^μ -ghost-vector meson vertex

FIGURE 3.7 Extra primitive divergences and associated renormalization constants to supplement those of Fig. 3.5.

$$\begin{aligned}\mathcal{L} = & -\tfrac{1}{4} Z_{3A} (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g Z_{1A} Z_{3A}^{-1} f^{abc} A^b_\mu A^c_\nu)^2 \\ & - \tfrac{1}{2\alpha} (\partial . A^a)^2 + \tfrac{1}{4} g^2 Z_{4A} f^{abc} A^b_\mu A^c_\nu f^{ab'c'} A^{b'\mu} A^{c'\nu} \\ & + Z_{3\chi} \bar{\chi}^a . \partial^\mu (\partial_\mu \delta^{ab} - g Z_{1\chi} Z_{3\chi}^{-1} f^{abc} A^c) \chi^b \\ & + Z_{3\phi} (\partial_\mu - i g Z_{1\phi} Z_{3\phi}^{-1} A^a_\mu T^a) \phi^+ . (\partial_\mu + i g Z_{1\phi} Z_{3\phi}^{-1} A^{b\mu} T^b) \phi \\ & - M^2 Z_M Z_{3\phi} \phi^+ \phi + Z_{3\psi} \bar{\psi} (i \not\partial + g Z_{1\psi} Z_{3\psi}^{-1} \not{A}^a T^a) \psi \\ & - m Z_m Z_{3\psi} \bar{\psi} \psi \\ & + I^{\mu a} Z_{3 I_\mu} \chi^a (\partial_\mu \delta^{ab} - g Z_{1 I_\mu} Z_{3 I_\mu}^{-1} f^{abc} A^c_\mu \chi^b) \\ & - \tfrac{1}{2} g Z_{1I} I^a f^{abc} \chi^b \chi^c \end{aligned} \qquad \ldots(3.107)$$

In (3.107), as well as the relations (3.69) there is also

$$\alpha_o = Z_{3A}\, \alpha \qquad \ldots(3.108)$$

The BRS identities (3.99)→ (3.103) now give

$$\left.\begin{aligned} Z_{3I_\mu} &= Z_{3\chi} \\ Z_{1I_\mu} &= Z_{1\chi} \\ Z_{1A} Z_{3\chi} - Z_{3A} Z_{1I_\mu} &= 0 \\ Z_{1I} &= Z_{1\chi} \\ \text{and} \qquad Z_{4A} &= 0 \end{aligned}\right\} \qquad \ldots(3.109)$$

By including matter fields one can also show the Taylor identity [33]

$$\frac{Z_{1A}}{Z_{3A}} = \frac{Z_{1\chi}}{Z_{3\chi}} = \frac{Z_{1\phi}}{Z_{3\phi}} = \frac{Z_{1\psi}}{Z_{3\psi}} \qquad \ldots(3.110)$$

The upshot of all this is that only Z_{3A}, $Z_{3\chi}$, $Z_{3\phi}$, $Z_{3\psi}$, Z_{1A}, $Z_{1\chi}$, $Z_{1\phi}$, $Z_{1\psi}$ are needed to renormalize all quantities.

Another point worth noting is that the quantity

$$g_o / g = Z_{1A} \; Z_{3A}^{-3/2} \qquad \ldots(3.111)$$

is gauge independent and source independent.

3.9 DIMENSIONAL REGULARIZATION

In the calculations to follow we need, of course, some way of formally defining the infinite quantities which appear. A particularly convenient choice of regularization scheme for non-abelian gauge theory is dimensional regularization (DR) [2,3,7,12,14,16,17,40]. This technique, developed in the early 1970's [41,42,43,44,45], is an improvement on analytic regularization and other schemes. Refs. [3,40] put DR in historical perspective.

The basic idea is to use the dimensionality of space-time as a complex parameter to regularize divergent integrals. For instance, a typical integral

$$I = \int \frac{d^4k}{(k^2-m^2)[(k+p)^2-m^2]} \qquad \ldots(3.112)$$

is badly defined in four dimensions, but is finite in two or three dimensional space-time by power counting. Therefore, the natural thing to do is generalize the number of dimensions from 4 to n, where $n=0,1,2,\ldots$; and then to replace n by a complex regulating parameter 2ℓ. Thus, the integral is analytically continued to regions of the ℓ plane where its existence is assured. Here the ultraviolet infinities show up as poles in gamma functions of ℓ. Denominators of integrals such as (3.112) are combined by means of Feynman parameters, and infrared infinities show up as end-point singularities in the parametric integral [3]. It is possible to carry out all calculations in a 2ℓ-dimensional space without complication so long as one remembers the assignments [40]:

$$\left.\begin{aligned} \eta^{\mu}{}_{\mu} &= 2\ell \\ p_\mu p_\nu &= \left(\frac{p^2}{2\ell}\right)\eta_{\mu\nu} \\ Tr(\gamma_\mu \gamma_\nu) &= 2^\ell \eta_{\mu\nu} \end{aligned}\right\} \qquad \ldots(3.113)$$

and any others which may explicitly involve the dimensionality of space-time.

The main advantages of DR are

1. It's inherent simplicity. Propagators are unaltered and

there is no need for the introduction of regulator fields and so on.

2. It respects the gauge invariance as established in Ward-Takahashi-Slavnov-Taylor identities.

3. With the assignments (3.113) all formal manipulations, like shifting of integration variable and symmetrization of the integrand, are allowed [2].

4. Certain highly divergent integrals can be formally regularized to zero. These are [2,3,17,22,40]:

   i) The lowest order massless tadpole, $\int \frac{d^{2\ell}k}{k^2}$.

   ii) The $\delta^4(0)$ term $\int d^{2\ell}k$.

   iii) Integrals over a polynomial, $\int d^{2\ell}k\,(k^2)^m$ where m =0, 1,2.... .

The dimensional regularization technique now rests on a firm basis. For this reason it is possible to approach its use in a somewhat 'cavalier' fashion in this thesis. That is, we generalize to 2ℓ dimensions precisely to make use of the above outlined advantages. Then at the end we replace 2ℓ by 4 and exhibit infinities as functions of the traditional ultraviolet and infrared cut-offs Λ and μ. There is a simple correspondence between this more transparent exposition of the infinities and the DR scheme's 'poles in ℓ' approach. This, and other calculationally expedient paths taken, will be outlined in the next chapter.

* * * * * * * * * *

<u>REFERENCES - CHAPTER 3</u>

1. R.A. Brandt : Ann. Phys. <u>52</u>, 122 (1969).

2. R. Gastmans : 'Gauge theories : an introduction' in *'Weak and Electromagnetic Interactions at High Energies,'* M. Lèvy, et al, eds., Plenum Press (N.Y.) (1976).

3. R. Delbourgo : Rep. Prog. Phys. <u>39</u>, 345 (1976).

4. R.A. Brandt and Ng Wing-Chiu : Phys. Rev. D10, 4198 (1974).

5. H. Pietschmann : 'Elementary introduction to gauge theories' in *'Facts and Prospects in Gauge Theories,'* P. Urban, ed., Springer-Verlag (N.Y.) (1978).

6. I.J.R. Aitchison : Oxford preprint 17/80 (1980).

7. G. Costa and M. Tonin : Revista del Nuovo Cim. 5, 29 (1975).

8. C.N. Yang and R.L. Mills : Phys. Rev. 96, 191 (1954).

9. B.W. Keck : 'Functional Methods,' lectures delivered at the University of Tasmania (1980), (to be published).

10. S. Coleman : 'Secret Symmetry : an introduction to spontaneous symmetry breakdown and gauge fields,' in *'Laws of Hadronic Matter,'* A. Zichichi, ed., Academic Press (N.Y.) (1975).

11. L. Faddeev : 'Introduction to functional methods' in *'Methods in Field Theory,'* R. Balian and J. Zinn-Justin, eds., Les Houches (1976).

12. B.W. Lee : 'Gauge theories,' ibid.

13. B.W. Lee : 'Path integrals and gauge transformations' in *'Renormalization and Invariance in Quantum Field Theory,'* E. Caianillo, ed., Plenum Press (N.Y.) (1974).

14. E.S. Abers and B.W. Lee : Phys. Rep. 9, 1 (1973).

15. W. Marciano and H. Pagels : Phys. Rep. 36, 137 (1978).

16. C. Nash " *'Relativistic Quantum Fields,'* Academic Press (Lond.) (1978).

17. J.C. Taylor : *'Gauge Theories of Weak Interactions,'* Camb. Univ. Press. (Camb.) (1976).

18. R. Delbourgo : 'Renormalization of Quantized Gauge Theories,' lectures delivered to the Australian Working Theory Summer School, Melbourne Univ. (1978), (unpublished).

19. J. Zinn-Justin : 'Renormalization of gauge theories' in *'Trends in Elementary Particle Physics,'* H. Rollnick and K. Dietz, eds., Springer-Verlag (Berlin), (1975).

20. W. Kummer : 'Renormalization of non-abelian gauge fields' in *'Current Problems in Elementary and Particle Physics,'* P. Urban, ed., Springer-Verlag (N.Y.) (1976).

21. L.D. Faddeev and V.N. Popov : Phys. Lett. B25, 29 (1967).

22. R. Delbourgo : 'Non-Abelian Gauge Theories,' lectures delivered at the University of Tasmania (1980) (unpublished).

23. D. Heckathorn : Nucl. Phys. B156, 328 (1979).

24. G. t'Hooft : Nucl. Phys. B33, 173 (1971).

25. B.W. Lee : Phys. Rev. D5, 823 (1972).

26. J. Frenkel : Phys. Rev. D13, 2325 (1976).

27. J. Frenkel and J.C. Taylor : Nucl. Phys. B109, 439 (1976).

28. T.P. Beven and R. Delbourgo : Lett. Nuovo Cim. 27, 565 (1980) and erratum, 28, 400 (1980).

29. T.P. Beven and R. Delbourgo : Lett. Nuovo Cim. 23, 433 (1978).

30. J.C. Ward : Phys. Rev. 78, 1824 (1950).

31. Y. Takahashi : Nuovo Cim. 6, 370 (1967).

32. A.A. Slavnov : Theor. Math. Phys. 10, 99 (1972).

33. J.C. Taylor : Nucl. Phys. B33, 436 (1971).

34. C. Becchi, A. Rouet and R. Stora : Commun. Math. Phys. 42, 127 (1975).

35. B.W. Lee and J. Zinn-Justin : Phys. Rev. D7, 1049 (1973).

36. R. Delbourgo : J. Phys. A11, 2057 (1978).

37. R. Delbourgo, A. Salam and J. Strathdee : Nuovo Cim. A23, 237 (1974).

38. W. Konetschny : Nuovo Cim. A44, 464 (1978).

39. J. Bernstein : Phys. Rev. D15, 2273 (1977).

40. G. Leibbrandt : Rev. Mod. Phys. 47, 849 (1975).

41. G. t'Hooft and M. Veltman : Nucl. Phys. B44, 189 (1972).

42. J. Ashmore : Lett. Nuovo Cim. 4, 289 (1972).

43. G.M. Cicuta and E. Montaldi : Lett. Nuovo Cim. 4, 329 (1972).

44. C.G. Bollini and J.J. Giambiagi : Nuovo Cim. 12B, 20 (1972).

45. E.R. Speer : J. Math. Phys. 15, 1 (1974).

4. RENORMALIZATION CONSTANTS IN THE COVARIANT, AXIAL AND COULOMB GAUGES.

In this chapter we present the wave-function and vertex renormalization constants to second order in the coupling constant g. Specific details of calculations can be found in Appendix G, to which frequent reference is made.

4.1 PREAMBLE

In reference [1] we presented a list of the wave-function renormalization constants in the axial gauge, and indicated a parallelism between these and those in the Lorentz covariant gauge with parameter $\kappa=-3$. This coincidence has been noticed by others [2,3]. Here we go through the computations in detail, as well as the same calculations for the Coulomb gauge. In the latter case there is a correspondence with the Fermi gauge, where $\kappa=1$. Since in the next chapter we will be re-deriving essentially the same results using a general propagator, a defence is in order: Much can be learnt, especially in the Coulomb gauge, from detailed calculations. For instance, whereas in Chapter 5 the only vertex renormalization constant explicitly calculated is $Z_{1\chi}$, here we obtain $Z_{1\phi}$, $Z_{1\psi}$ and $Z_{1\chi}$ in all three gauges. We are thus able to confirm the Ward identities of Section 3.8 between the renormalization constants to this order. As well as this, transversality of the vector meson self-energy is explicitly established.

The Feynman rules for the covariant (c), axial (a) and Coulomb or radiation (r) gauges have been detailed in Figures 3.1, 3.2 and 3.3 respectively. The axis n_μ in non-covariant gauges is always taken to be unit timelike. Of interest is the structure of the vector gauge meson propagator in each case, written in a transparent matrix notation as:

Covariant gauges

$$\Delta_{(c)\mu\nu}(k) = \frac{1}{k^2}\begin{bmatrix} \frac{-ck_0^2+k^2}{k^2} & (1-c)\frac{k_0 k_j}{k^2} \\ (1-c)\frac{k_i k_0}{k^2} & \delta_{ij}+(1-c)\frac{k_i k_j}{k^2} \end{bmatrix} \quad \ldots(4.1c)$$

Axial gauge

$$\Delta_{(a)\mu\nu}(k) = \frac{1}{k^2}\begin{bmatrix} 0 & 0 \\ 0 & \delta_{ij} - \frac{k_i k_j}{k_0^2} \end{bmatrix} \quad \ldots(4.1a)$$

Coulomb gauge

$$\Delta_{(r)\mu\nu}(k) = \frac{1}{k^2}\begin{bmatrix} \frac{k^2}{\underline{k}^2} & 0 \\ 0 & \delta_{ij} - \frac{k_i k_j}{\underline{k}^2} \end{bmatrix} \quad \ldots(4.1r)$$

where $i, j = 1,2,3$ label rows and columns respectively.

Immediately, the following points are apparent: Covariant gauges involve non-physical scalar and longitudinal gauge mesons (see Section 2.3 in this regard). But of course the self-energies and vertex corrections will contain covariant kinematic factors, leading to a covariant renormalization procedure. The axial gauge, on the other hand, has a purely transverse Yang-Mills propagator. But it turns out that in this case the renormalization procedure is covariant too [3]. In the Coulomb gauge, care must be taken. Indeed, here there are two distinct types of virtual quanta able to be exchanged - longitudinal mesons, with $\Delta_{00}(k) = \frac{1}{\underline{k}^2}$, and transverse mesons, with $\Delta_{ij}(k) = \frac{1}{k^2}(\delta_{ij} - \frac{k_i k_j}{\underline{k}^2})$ [4,5]. The former are associated with the static Coulomb interaction. Clearly, in this gauge the vector meson self-energy, and all vertex corrections, must be resolved into separate longitudinal and transverse pieces, and renormalizations performed accordingly. That is, for the Yang-

Mills field, for instance, the renormalized field is obtained from the bare one by rescaling [4].

$$\left.\begin{aligned} A_{\mu\,0} &= Z^{1/2}_{3A,\mu\nu}\; A^{\nu} \\ \text{with}\quad Z^{1/2}_{3A,\mu\nu} &= (Z^{\parallel}_{3A})^{1/2}\,\frac{n_\mu n_\nu}{n^2} + (Z^{\perp}_{3A})^{1/2}\,\eta^{\perp}_{\mu\nu} \\ \text{where}\quad \eta^{\perp}_{\mu\nu} &= \eta_{\mu\nu} - \frac{n_\mu n_\nu}{n^2} \end{aligned}\right\} \quad \ldots(4.2)$$

The decomposition of the various Z into parts that are parallel and orthogonal to n_μ is given in Figure 4.1.

In covariant gauges of course, $Z^{\parallel} = Z^{\perp}$ for all Z. In axial gauges, the same is true. This is due to the complete absence of ghosts in this gauge. In the Coulomb gauge, the ghosts decouple from the longitudinal component of the Yang-Mills propatagor only, and so $Z^{\parallel} \neq Z^{\perp}$. Nevertheless, we shall see that this decoupling does lead to some rather simple and interesting results for $Z^{\parallel}$.

Since Chapter 5 reproduces the results of this chapter, I have chosen here to ultimately place both scalar and fermion fields in the fundamental (quark) representation (see Appendix B, especially equations (B9) to (B18)). The reason for this is that it enables the reader to find in Table 4.11 at the end of this chapter, the various Z for any value of n. Later, in Table 5.1, he may place these fields in whatever representation he pleases and obtain the corresponding results. Also, it is interesting to observe the rather fortuitous combination of factors of n which give covariant results for the axial gauge corrections.

In Appendix G, the masses of both scalar mesons and fermions are written as m, there being no need to make the distinction as was done in Chapter 3. The loop integrals over virtual momenta are written as $\int d^{2l}k\ldots\ldots$, to indicate where

LONGITUDINAL AND TRANSVERSE PARTS OF RENORMALIZATION CONSTANTS

Correction   Assoc.   Correction   Assoc.

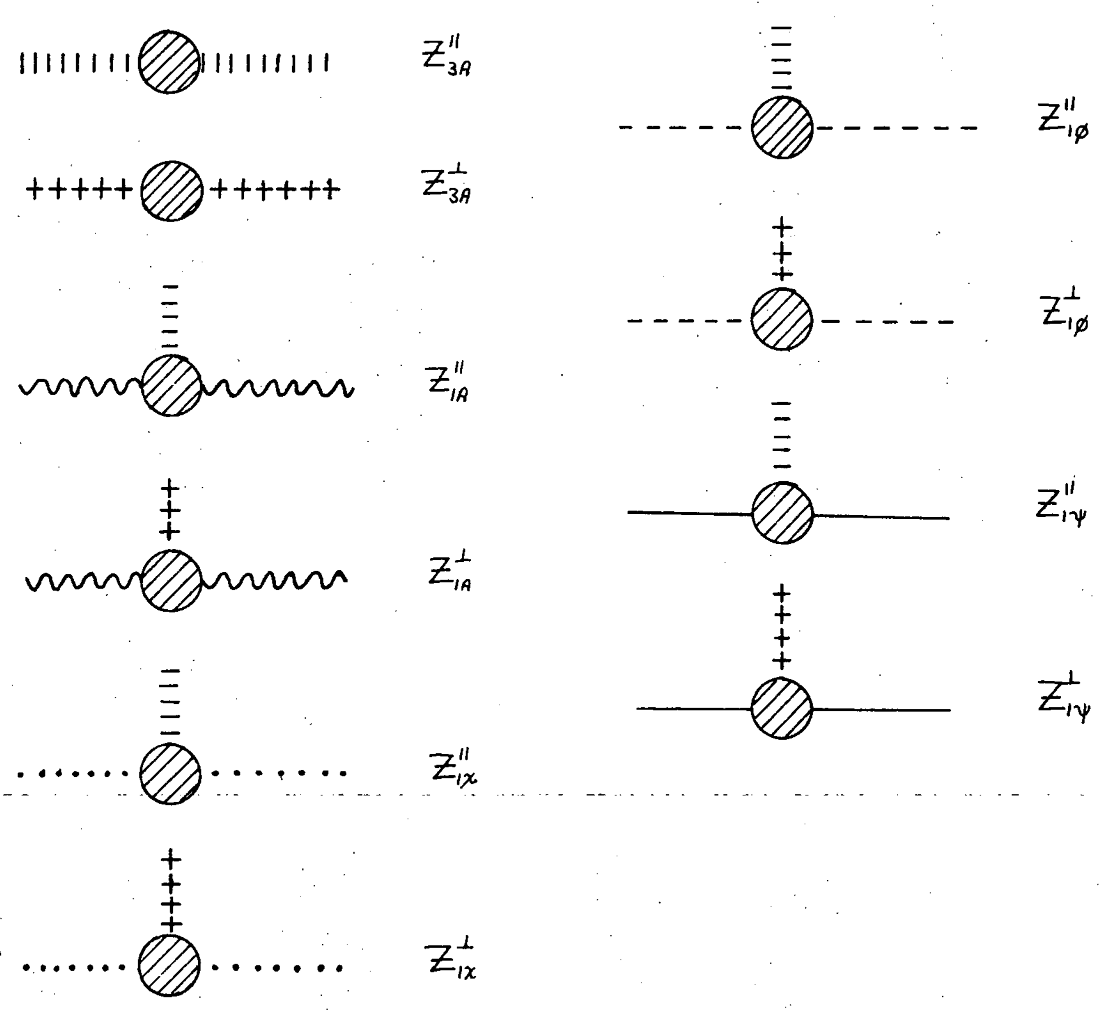
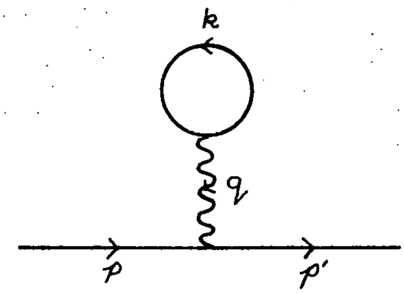


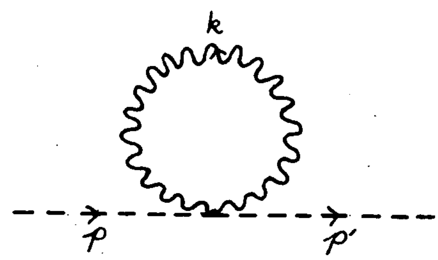
FIGURE 4.1 Resolution of self-energies and vertex corrections into longitudinal and transverse pieces, with associated renormalization constants Z. |||||| and ++++ label the longitudinal and transverse components of the vector meson propagator respectively. Other Z retain the same form as in Fig. 3.5.

dimensional regularization is being used. It is only in the final step(s) that the limit $\ell \to 4$ is taken and final integrations performed as outlined in Appendix D.

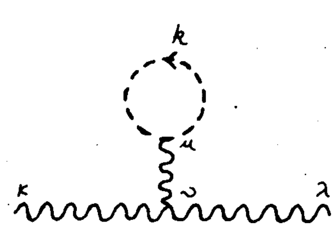
The term 'tadpole' diagram refers to one of the form, for example



whilst 'seagull' diagram denotes one like



It turns out that seagull diagrams give formally zero divergent contribution in the dimensional regularization scheme [6], as stated in Section 3.9. As an example, this is shown explicitly in Appendix G for the scalar meson self-energy in the covariant gauge. Tadpole terms vanish by reasons of Lorentz invariance, as can easily be seen for the ghost tadpole in the Yang-Mills self-energy, viz:



which has the form $\Delta_{\kappa\lambda\nu}\Delta^{\mu\nu}(0)\int \frac{k_\mu d^4k}{k^2}$. Thus, although seagull diagrams are included in the figures relating to each calculation for completeness' sake, they have no effect on the divergent part. Tadpole diagrams will be ignored.

To extract the relevant ultraviolet infinities it is necessary to expand the matrix elements in terms of some external momentum, say p. After the use of dimensional regularization, the surviving integrals are logarithmically divergent. The labels I_1 etc. used in Appendix G refer to these integrals which are worked out in Appendix D and enumerated in Table D1. They are given as multiples of the constant $L = \frac{g^2}{16\pi^2} \ln \frac{\Lambda^2}{\mu^2}$. Of course, it is necessary to use a principal value prescription [7,8] to deal with the singularities of (4.1a) in the k_0 plane so as to preserve unitarity on mass shell (see Appendix D). Unlike certain authors,[9, 10], we do not invoke any infrared - ultraviolet cancellation with regard to these integrals.

At this point I mention one notational liberty which has been taken - the symbols $\Pi(p^2)$, $\Sigma(p)$ etc., are used to label the various diagrams relating to each calculation, as well as the overall result. A seagull or tadpole diagram not so labelled is not considered in further work.

The order in which computation proceeds has no special significance. Inverse propagators and proper scattering vertices come from Table 3.6

The S-matrix being written as

$$S = 1 + i\,T \qquad \ldots(4.3)$$

leads to the following assignments for each self-energy/vertex correction:

i for each vertex and propagator,

-1 for each closed fermion or ghost loop, and an overall factor of $-i$.

Group combinatorical factors for each relevant diagram are listed in equations (G1) to (G21).

4.2 EVALUATION OF RENORMALIZATION CONSTANTS

SCALAR MESON WAVE-FUNCTION

The complete inverse propagator for the scalar meson is

$$\Delta_\phi'^{-1}(p) = Z_{3\phi}\,(p^2-m_0^2) + \Pi(p^2) \qquad \ldots(4.4)$$

Hence the wave-function renormalization constant $Z_{3\phi}|_\infty$ is given by

$$Z_{3\phi}|_\infty = 1 - \left.\frac{\partial\Pi(p^2)|_\infty}{\partial p^2}\right|_{p^2=m^2} \qquad \ldots(4.5)$$

The diagrams contributing to the second order scalar meson self-energy are given in Figure 4.2. From these, the general form of $\Pi(p^2)$ is found to be:

$$\Pi(p^2) = -i\int d^{2l}k\,\Big[i\Lambda^a_\phi(p,p+k)\,i\Delta^{ab,\mu\nu}(k)\,i\Delta_\phi(p+k)\,i\Lambda^b_{\phi\nu}(p+k,p) + i\Lambda^{ab}_{\phi\,\mu\nu}\,i\Delta^{ab,\mu\nu}(k)\Big]$$
$$+ \text{ tadpoles} \qquad \ldots(4.6)$$

$\Pi_{(ii)}(p^2)$ gives zero divergent contribution, as is shown for the covariant gauge case in (G23). Thus the specific formulae for $\Pi(p^2)|_\infty$ in the covariant, axial and Coulomb gauges appear in (G24), (G59) and (G96) respectively. Of course, $\Pi(p^2)|_\infty$ is covariant in covariant gauges (G25). We also find it to be so in the axial gauge (G60). In the Coulomb gauge the separate contributions from the virtual longitudinal and transverse quanta, (G97) and (G98), combine to give a covariant result (G99). To summarise, then, the results for the three cases are:

$$\Pi(p^2)|_\infty = \begin{cases} [(\alpha-3)p^2-m^2]\,C_\phi L & \ldots(4.7c) \\ (-6p^2+3m^2)\,C_\phi L & \ldots(4.7a) \\ -(2p^2+m^2)\,C_\phi L & \ldots(4.7r) \end{cases}$$

yielding via (4.5) the wave-function renormalization constants:

SCALAR MESON SELF-ENERGY

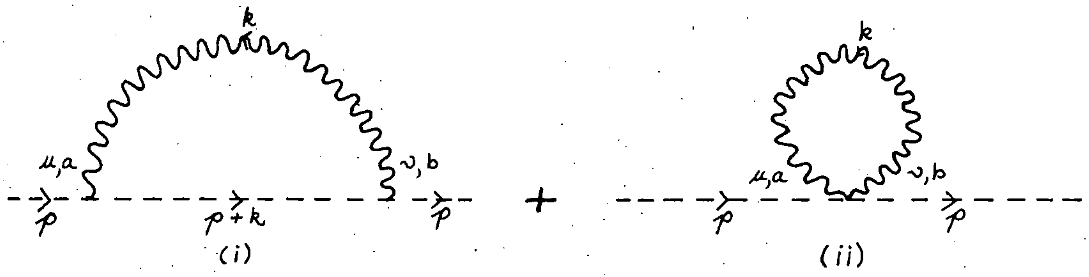


FIGURE 4.2 Diagrams contributing to the second order scalar meson self-energy. They yield $Z_{3\phi}$.

FERMION SELF-ENERGY

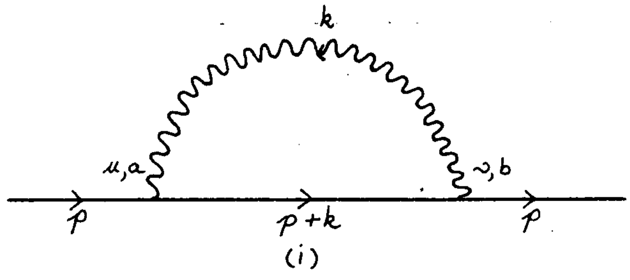


FIGURE 4.3 The diagram contributing to the second order fermion self-energy. It yields $Z_{3\psi}$.

$$Z_{3\phi}\Big|_\infty = \begin{cases} 1-(c-3)C_\phi L = 1-\frac{(c-3)(n^2-1)}{2n}L & \text{...(4.8c)} \\ 1+6C_\phi L = 1+\frac{3(n^2-1)}{2n}L & \text{...(4.8a)} \\ 1+2C_\phi L = 1+\frac{n^2-1}{2n}L & \text{...(4.8r)} \end{cases}$$

FERMION WAVE FUNCTION

The complete inverse propatagor for the fermion is

$$\Delta'^{-1}_\psi(p) = Z_{3\psi}(\not{p}-m_0)+\Sigma(p) \qquad \text{...(4.9)}$$

Hence the wave-function renormalization constant $Z_{3\psi}|_\infty$ is given by

$$Z_{3\psi}\Big|_\infty = 1-\frac{\partial\Sigma(p)|_\infty}{\partial\not{p}}\Bigg|_{\not{p}=m_0} \qquad \text{...(4.10)}$$

The diagrams contributing to the second order fermion self-energy are given in Figure 4.3. From these, the general form of $\Sigma(p)$ is found to be

$$\Sigma(p) = -i\int d^{2\omega}k\, i\Lambda^a_{\psi\mu}\, i\Delta^{ab,\mu\nu}(k)\, i\Delta_\psi(p+k)\, i\Lambda^b_{\psi\nu}$$

$$+ \text{ tadpole} \qquad \text{...(4.11)}$$

The specific formulae for $\Sigma(p)|_\infty$ in the covariant, axial and Coulomb gauges appear in (G26), (G61) and (G100) respectively. Once again, it is easy to show that $\Sigma(p)|_\infty$ is covariant in the covariant, (G27), and axial, (G62), gauges. And the separate longitudinal and transverse contributions, (G101) and (G102), to the Coulomb gauge case both combine as before to give a covariant result (G103). In summary, the results for the three cases are:

$$\Sigma(p)\Big|_\infty = \begin{cases} [c\not{p}-(3+c)m]C_\psi L & \text{...(4.12c)} \\ -3\not{p}\,C_\psi L & \text{...(4.12a)} \\ (\not{p}+4m)\,C_\psi L & \text{...(4.12r)} \end{cases}$$

yielding from (4.10) the wave-function renormalization constants :

$$Z_{3\psi}\Big|_\infty = \begin{cases} 1 - \varepsilon C_\psi L = 1 - \frac{\varepsilon(n^2-1)}{2n} L & \ldots(4.13c) \\ 1 + 3C_\psi L = 1 + \frac{3(n^2-1)}{2n} L & \ldots(4.13a) \\ 1 - C_\psi L = 1 - \frac{n^2-1}{2n} L & \ldots(4.13r) \end{cases}$$

VECTOR GAUGE MESON SELF-ENERGY

The complete inverse propagator in covariant gauges for the vector meson is

$$\Delta^{aa'\,\prime -1}_{\kappa\kappa'}(p) = Z_{3A}\,\delta^{aa'}(-\eta_{\kappa\kappa'}p^2 + p_\kappa p_{\kappa'}) + \Pi^{aa'}_{\kappa\kappa'}(p) \quad \ldots(4.14)$$

The self energy, $\Pi^{aa'}_{\kappa\kappa'}(p)$, involves not only contributions from the self-interaction of the vector meson itself, but also from the scalar meson and fermions. Vector meson, scalar meson and fermion diagrams contributing to the second order vector meson self-energy are given in Figures 4.4, 4.5 and 4.6 respectively. Since the last two give gauge independent results, we deal with the pure Yang-Mills part first. The general form of $\Pi^{aa'}_{\kappa\kappa'}(p)$ due to the pure Yang-Mills and ghost field interactions is found from Figure 4.4 to be:

$$\begin{aligned}\Pi^{aa'}_{\kappa\kappa'}(p) = -i\int d^{2\ell}q\, d^{2\ell}r\; \delta(p+q+r)\,\times \\ \times\big[\tfrac{1}{2}\, i\Lambda^{abc}_{\kappa\lambda\mu}(p,q,r)\, i\Delta^{bb',\lambda\lambda'}(q)\, i\Delta^{cc',\mu\mu'}(r)\, i\Lambda^{a'b'c'}_{\kappa'\lambda'\mu'}(-p,-q,-r) \\ - i\Lambda^{abc}_{\chi\;\kappa}(q)\, i\Delta^{bb'}_{\chi}(q)\, i\Delta^{cc'}_{\chi}(r)\, i\Delta^{a'b'c'}_{\chi\;\kappa'}(r)\,\big]\end{aligned}$$

$$\ldots(4.15)$$

The specific formulae for $\Pi^{aa'}_{\kappa\kappa'}(p)\big|_\infty$ in the covariant, axial and Coulomb gauges appear in equations (G28), (G63) and (G104) respectively. As a warm-up exercise in each case, the doubly transverse nature of the self-energy as predicted by the Slavnov-Taylor identities in equation (3.86) is explicitly verified in equations (G29) to (G30), (G64) to (G67) and (G105) to (G109). This means that, in complete generality, the self-energy may be decomposed as

$$\begin{aligned}\Pi_{\kappa\kappa'}(p)\Big|_\infty = (-\eta_{\kappa\kappa'}p^2 + p_\kappa p_{\kappa'})\Pi_C + (p^2\eta_{\kappa 0} - p_0 p_\kappa)(p^2\eta_{\kappa' 0} - p_0 p_{\kappa'})\Pi_N \\ + \big[(p^2\eta_{\kappa 0} - p_\kappa p_0)\eta_{\kappa' 0} + (p^2\eta_{\kappa' 0} - p_{\kappa'} p_0)\eta_{\kappa 0}\big]\Pi_M \quad \ldots(4.16)\end{aligned}$$

PURE VECTOR MESON SELF-ENERGY

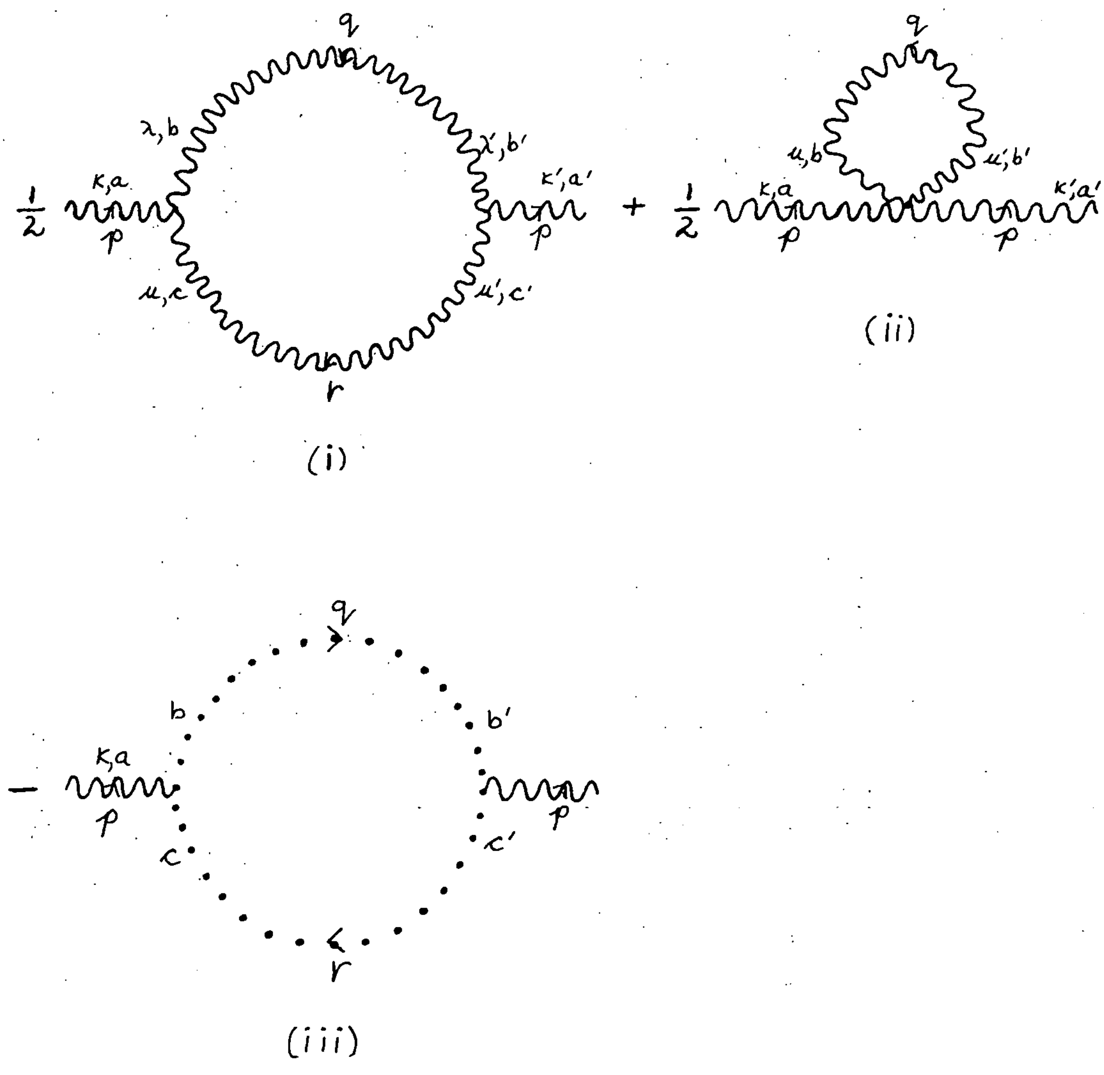


FIGURE 4.4 Diagrams due to Yang-Mills self interactions and the Yang-Mills-ghost interaction which contribute to the second order gauge vector meson self-energy. They yield Z_{3A}.

for a unit timelike axis n_μ, where the kinematic factors Π_C, Π_M and Π_N are scalar functions of p^2 and p_0. Obviously, in covariant gauges Π_N and Π_M are absent. By power counting, Π_N is finite and thus it is sufficient to find the infinite parts of Π_C and Π_M for the axial and Coulomb gauge renormalizations. It turns out that Π_M is absent in the axial gauge [3,4,7]. It is only in the Coulomb gauge that one requires both Π_C and Π_M, and here of course, as noted in equation (4.2), the wave-function renormalization constant Z_{3A} will be split into parts parallel and orthogonal to n_μ.

The results for the covariant, axial and Coulomb gauges can be found in (G35), (G75), and (G125) and (G126) respectively. They are:

$$\Pi_C = \begin{cases} -\frac{(13-3c)}{6} C_A L = -\frac{(13-3c)n}{6} L & \ldots(4.17c) \\ -\frac{11}{3} C_A L = -\frac{11n}{3} L & \ldots(4.17a) \\ -C_A L = -n L & \ldots(4.17r) \end{cases}$$

and

$$\Pi_M = \begin{cases} 0 & \ldots(4.18c) \\ 0 & \ldots(4.18a) \\ \frac{4}{3} C L = \frac{4n}{3} L & \ldots(4.18r) \end{cases}$$

When $n_\mu = (1,0,0,0)$, the parts of $\Pi_{kk'}(p)\big|_\infty$ in (4.16) which are wholly parallel, or wholly perpendicular, to n_μ are

$$\Pi_{00}(p)\big|_\infty = \not{p}^2 \, (\Pi_C - 2\Pi_M)$$

and

$$\Pi_{kk}(p)\big|_\infty = (3\,p^2 + \not{p}^2)\,\Pi_C$$

respectively. Thus, the wave-function renormalization constants $Z_{3A}^{\parallel}$ and $Z_{3A}^{\perp}$ are given by

$$\left.\begin{aligned} Z_{3A}^{\parallel} &= 1 - \Pi_C + 2\Pi_M \\ Z_{3A}^{\perp} &= 1 - \Pi_C \end{aligned}\right\} \qquad \ldots(4.19)$$

Recall that in the covariant and axial gauges, $Z_{3A}^{\parallel} = Z_{3A}^{\perp}$.

Hence, we arrive at the results:

$$Z_{3A}\Big|_{\infty} = \begin{cases} 1 + \frac{(13-3c)}{6} C_A L = 1 + \frac{(13-3c)n}{6} L & \ldots(4.20c) \\[2ex] 1 + \frac{11}{3} C_A L = 1 + \frac{11n}{3} L & \ldots(4.20a) \\[2ex] \left.\begin{aligned} (Z_{3A}^{\parallel}) \quad & 1 + \tfrac{11}{3} C_A L = 1 + \tfrac{11n}{3} L \\ (Z_{3A}^{\perp}) \quad & 1 + C_A L = 1 + n L \end{aligned}\right\} & \ldots(4.20r) \end{cases}$$

It is worth noting that in the Coulomb gauge calculations leading to (4.20v) not only does the ghost field make no contribution but also the <u>completely</u> longitudinal self-energy term is absent (see equations (G119) and (G122) in this regard). Also, the longitudinal-transverse pair self-energy terms give negative quantities in (G115) and (G119); whilst the purely transverse terms give positive quantities in (G116) and (G121). The emission and absorption of a virtual pair of Coulomb gauge quanta comprising one longitudinal and one transverse meson has been linked with asymptotic freedom [4].

Z_{3A} of (4.20) must be supplemented by terms coming from the scalar meson and fermion loops of Figures 4.5 and 4.6. The general forms of these additions to the self-energy are:

$$\tilde{\Pi}^{aa'}_{\kappa\kappa'}(p) = -i\int d^{2\ell}k \; i\Lambda^{a}_{\phi\kappa}(p, p+k)\, i\Delta_{\phi}(k)\, i\Delta_{\phi}(p+k)\, i\Lambda^{a'}_{\phi\kappa}(p+k, p)$$

$$+ \text{ seagull and tadpole} \qquad \ldots(4.21)$$

$$\Sigma^{aa'}_{\kappa\kappa'}(p) = -i\int d^{2\ell}k \; i\Lambda^{a}_{\psi\kappa}\, i\Delta_{\psi}(k)\, i\Delta_{\psi}(p+k)\, i\Lambda^{a'}_{\psi\kappa'}$$

$$+ \text{ tadpole} \qquad \ldots(4.22)$$

Specific formulae appear in (G36) and (G38).

SCALAR FIELD DIAGRAMS APPEARING IN THE

VECTOR MESON SELF-ENERGY

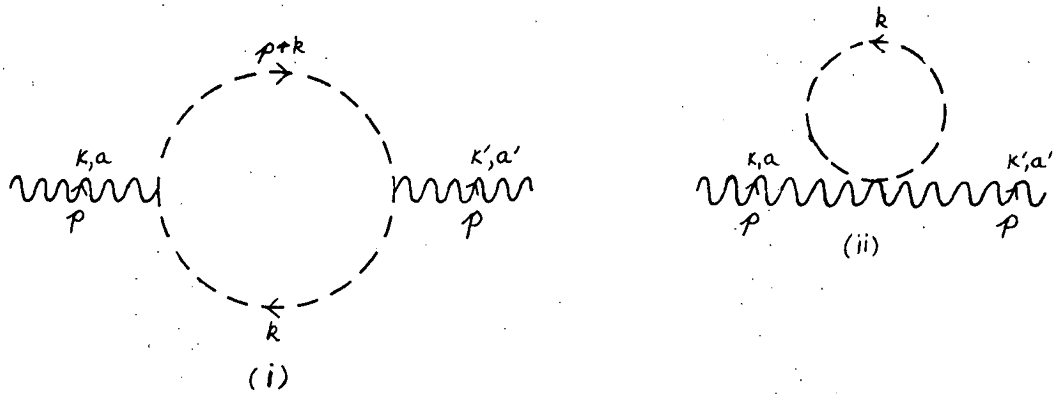


FIGURE 4.5 The two diagrams due to the scalar field which contribute to the second order Yang-Mills self-energy. They give an additional term in Z_{3A}.

SPINOR FIELD DIAGRAM APPEARING IN THE

VECTOR MESON SELF-ENERGY

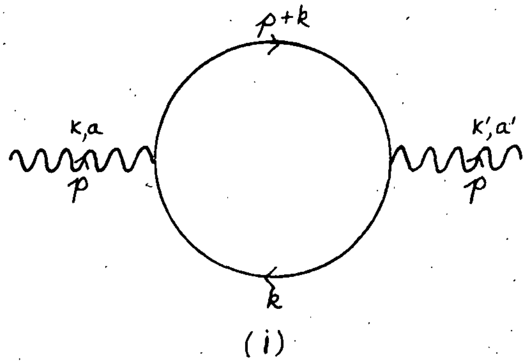


FIGURE 4.6 The diagram due to the spinor field which contributes to the second order Yang-Mills self-energy. It gives an additional term in Z_{3A}.

The resulting additional factors, which of course are gauge-independent, are also found to be doubly transverse, in confirmation of (3.86). They are given in (G37) and (G39) as

$$\tilde{\pi}^{aa'}_{\kappa\kappa'}(p)\Big|_\infty = \delta^{aa'}[-\eta_{\kappa\kappa'}p^2 + p_\kappa p_{\kappa'}]\ \tfrac{1}{3}\ T_\phi L \qquad \ldots(4.23)$$

and

$$\Sigma^{aa'}_{\kappa\kappa'}(p)\Big|_\infty = \delta^{aa'}[-\eta_{\kappa\kappa'}\, p^2 + p_\kappa\, p_{\kappa'}]\ \tfrac{4}{3}\ T_\psi L \qquad \ldots(4.24)$$

where T_ϕ and T_ψ are the normalizations of the structure constants in the representations chosen for the scalar mesons and fermions respectively. In the fundamental representation, $T_\phi = T_\psi = \frac{1}{2}$. From (4.16), (4.23) and (4.24) it is apparent that $\tilde{\pi}^{aa'}_{\kappa\kappa'}(p)\Big|_\infty$ and $\Sigma^{aa'}_{\kappa\kappa'}(p)\Big|_\infty$ simply supplement the coefficient π_c determined in (4.17) and thus appear in Z_{3A} of (4.20) via (4.19). For instance, in covariant gauges

$$Z_{3A}\Big|_\infty = 1 - \frac{(13-3c)}{6} C_A L - \tfrac{1}{3} T_\phi L - \tfrac{4}{3} T_\psi L \qquad \ldots(4.25)$$

and so on.

SCALAR VERTEX

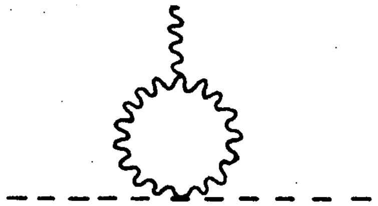
The proper scattering vertex for scalar mesons is

$$g^{-1}\Lambda^{a\,\prime}_{\phi\lambda}(p,p') = T^a(p+p')_\lambda\, Z_{1\phi} + \Gamma^a_\lambda(p) \qquad \ldots(4.26)$$

Hence the vertex renormalization constant $Z_{1\phi}$ is given by

$$Z_{1\phi} - 1 = \textit{coefficient of } T^a(p+p')_\lambda \textit{ in } \Gamma^a_\lambda(p) \qquad \ldots(4.27)$$

The diagrams contributing to the second order scalar vertex correction are listed in Figure 4.7. It is noteworthy that the third diagram,



has a vanishing combinatorical coefficient as shown in (G13). Therefore, it takes no part in the subsequent renormalization. So the general form of the vertex correction is:

SCALAR VERTEX CORRECTION

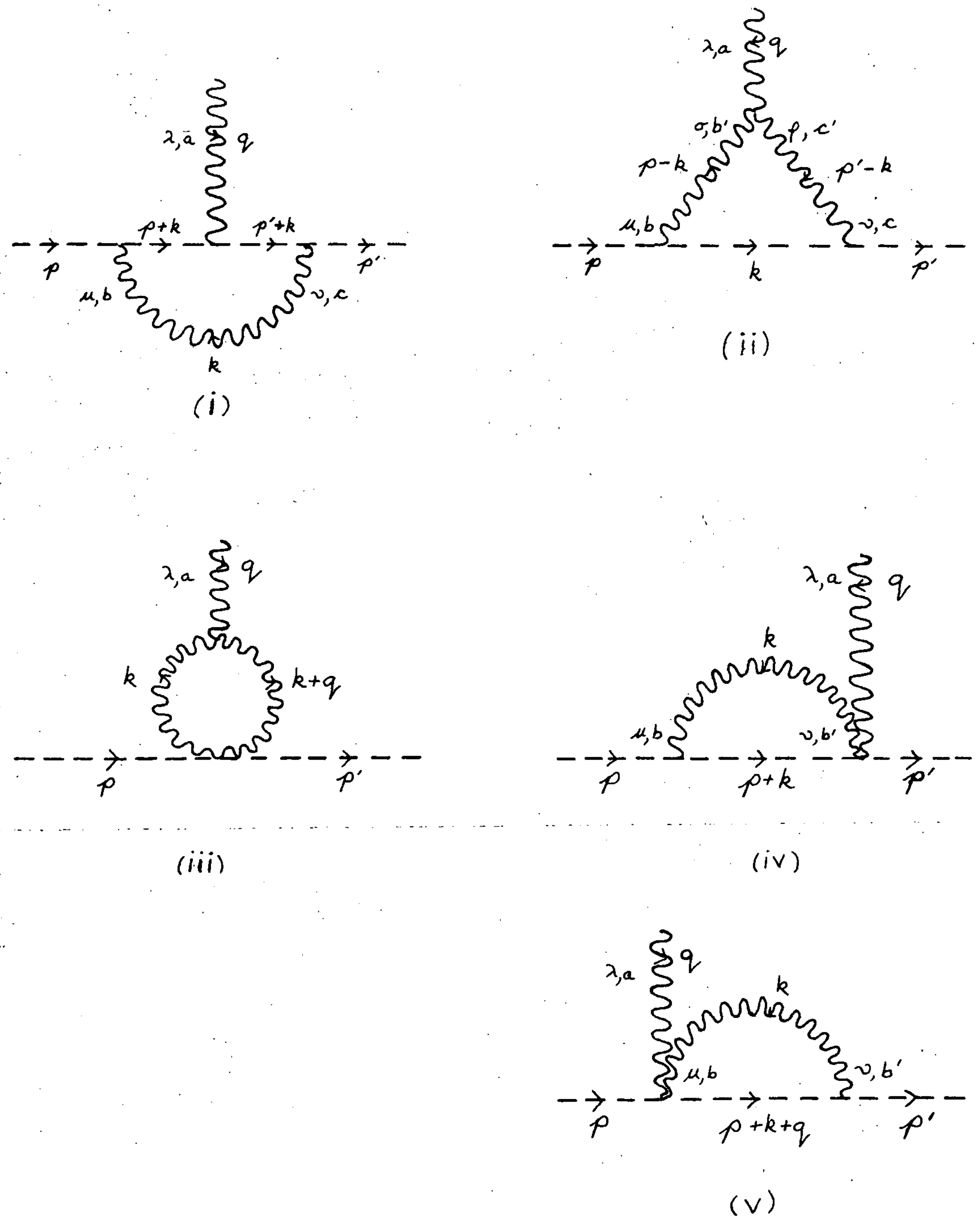


FIGURE 4.7 Diagrams contributing to the second order scalar-vertex correction. Diagram (iii) has zero combinatorical coefficient. They yield $Z_{1\phi}$.

$$\Gamma^a_\lambda(p) = -i \int d^{2l}k \{ i\Lambda^b_{\phi\mu}(p,p+k)\, i\Delta_\phi(p+k)\, i\Lambda^a_{\phi\lambda}(p+k,p'+k) \times$$
$$\times\, i\Delta_\phi(p'+k)\, i\Delta^{bc,\mu\nu}(k)\, i\Delta^c_{\phi\nu}(p'+k,p')$$

$$+ i\Lambda^b_{\phi\mu}(p,k)\, i\Delta^{bb',\mu\sigma}(p-k)\, i\Lambda^{ab'c'}_{\lambda\sigma\rho}(q,p-k,p'-k) \times$$
$$\times\, i\Lambda^{c'c,\rho\nu}(p'-k)\, i\Delta_\phi(k)\, i\Lambda^c_{\phi\nu}(k,p)$$

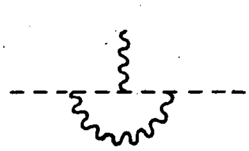
$$+ i\Lambda^b_{\phi\mu}(p,p+k)\, i\Delta_\phi(p+k)\, i\Delta^{bb',\mu\nu}(k)\, i\Lambda^{b'a}_{\phi\nu,\lambda}$$

$$+ i\Lambda^{ab}_{\phi\lambda\mu}\, i\Delta_\phi(p'+k)\, i\Delta^{bb',\mu\nu}(k)\, i\Lambda^{b'}_{\phi\nu}(p'+k,p') \} \quad \ldots (4.28)$$

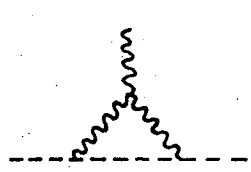
Specific formulae for $\Gamma^a_\lambda(p)\big|_\infty$ in the covariant, axial and Coulomb gauges for each diagram appear in (G40), (G42), (G44); (G76), (G79), (G83); and (G127), (G136), and (G145) respectively.

We are presented here with notational complications. $\Gamma^a_\lambda(p)$ will denote a covariant total vertex correction. In the non-covariant gauges this is separated into parts parallel and orthogonal to the axis n_μ. Since n_μ is unit timelike, these are denoted $\Gamma^a_0(p)$ and $\Gamma^a_l(p)$ respectively. In the Coulomb gauge, virtual longitudinal and transverse quanta are involved. These are labelled in turn by L and T. And all this is done diagram by diagram in Appendix G. There the reader may find any particular result he pleases. Here we summarize the major details.

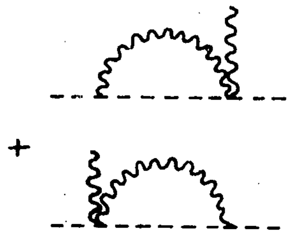
In *covariant* gauges, each diagram gives a separately covariant result. These are, from (G41), (G43) and (G45):



$$\Gamma^a_{\lambda(i)}(p)\big|_\infty = -2p_\lambda T^a \frac{c}{2n} L$$



$$\Gamma^a_{\lambda(ii)}(p)\big|_\infty = 2p_\lambda T^a \frac{3cn}{4} L$$

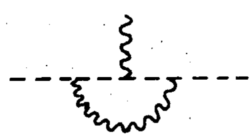


$$\Gamma^a_{\lambda\,(iv)+(v)}(p)\Big|_\infty = -2p_\lambda T^a \frac{3(n^2-2)}{4n} L$$

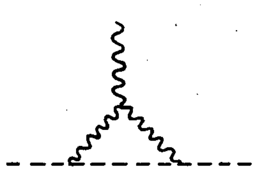
Whence, from (G46b):

$$\Gamma^a_\lambda(p)\Big|_\infty = 2p_\lambda T^a \left[\frac{3(\alpha-1)n^2-2\alpha+6}{4n}\right] L \qquad \ldots(4.29c)$$

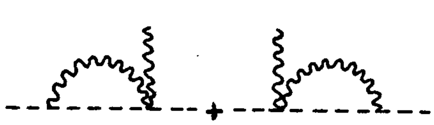
In the *axial* gauge, whilst the diagrams are each separately non-covariant, they add to produce a covariant total [3]. When the external field is timelike (resp. spacelike), we obtain the longitudinal (resp. transverse) contribution to $Z_{1\phi}$. From (G77), (G78), (G80), (G81), (G82) and (G83) the separate terms adding to the vertex correction are:



$$\Gamma^a_{o(i)}(p)\Big|_\infty = 2p_o T^a \tfrac{3}{n} L$$
$$\Gamma^a_{\ell(i)}(p)\Big|_\infty = 2p_\ell T^a \tfrac{1}{n} L$$



$$\Gamma^a_{o(ii)}(p)\Big|_\infty = -2p_o T^a 3n L$$
$$\Gamma^a_{\ell(ii)}(p)\Big|_\infty = -2p_\ell T^a 2n L$$

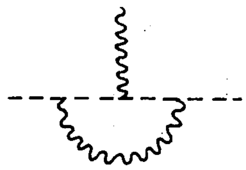


$$\Gamma^a_{o\,(iv)+(v)}(p)\Big|_\infty = 0$$
$$\Gamma^a_{\ell\,(iv)+(v)}(p)\Big|_\infty = -2p_\ell T^a \frac{n^2-2}{n} L$$

Whence, from (G84b), (G85b) and (G86):

$$\Gamma^a_\lambda(p)\Big|_\infty = \Gamma^a_o(p)\Big|_\infty = \Gamma^a_\ell(p)\Big|_\infty$$
$$= 2p_\lambda T^a \left[\frac{-3(n^2-1)}{n}\right] L \qquad \ldots(4.29a)$$

The *Coulomb* gauge exhibits an interesting situation. The total correction is non-covariant, leading to separate parallel and orthogonal renormalizations. It is worth noting that in some diagrams the divergent part is derived solely from that sub-diagram involving either a longitudinal or transverse quantum, the other sub-diagram giving zero infinite part. We comment on this where applicable. From (G131), (G135), (G140), (G144), (G148) and (G150) the separate terms adding to the vertex correction are:

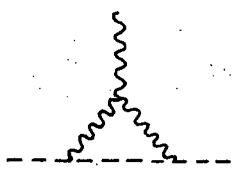


$$\Gamma^a_{o(i)}(p) = -2p_o T^a \tfrac{1}{n} L$$

(virtual quantum longitudinal)

$$\Gamma^a_{\ell(i)}(p) = -2p_\ell T^a \tfrac{1}{3n} L$$

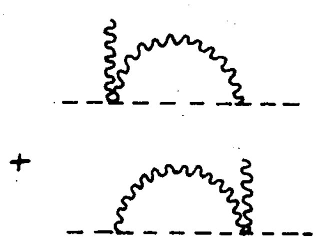
(virtual quantum transverse)



$$\Gamma^a_{o(ii)}(p)\Big|_\infty = 0$$

$$\Gamma^a_{\ell(ii)}(p)\Big|_\infty = 2p_\ell T^a n L$$

(the sub-diagram involving a pair of virtual transverse quanta is absent)



$$\Gamma^a_{o(iv)+(v)}(p)\Big|_\infty = -2p_o T^a \tfrac{n^2-2}{n} L$$

(virtual quantum longitudinal)

$$\Gamma^a_{\ell(iv)+(v)}(p)\Big|_\infty = -2p_\ell T \, \frac{2(n^2-2)}{3n} L$$

(virtual quantum transverse)

Whence, from (G152b) and (G153b):

$$\left.\begin{aligned} \Gamma^a_o(p)\Big|_\infty &= 2p_o T^a \left[\tfrac{1-n^2}{n}\right] L \\ \Gamma^a_\ell(p)\Big|_\infty &= 2p_\ell T^a \left[\tfrac{n^2+3}{n}\right] L \end{aligned}\right\} \qquad \ldots(4.29r)$$

Thus, using (4.27) we arrive at the results:

$$Z_{1\phi}\Big|_\infty = \begin{cases} 1 - \left[\dfrac{3(c-1)n^2-2c+6}{4n}\right] L & \ldots(4.30c) \\[2ex] 1 + \dfrac{3(n^2-1)}{n} L & \ldots(4.30a) \\[2ex] \left.\begin{aligned} (Z^{\parallel}_{1\phi}) \quad & 1 + \frac{n^2-1}{n} L \\ (Z^{\perp}_{1\phi}) \quad & 1 - \frac{n^2+3}{3n} L \end{aligned}\right\} & \ldots(4.30r) \end{cases}$$

FERMION VERTEX

The proper scattering vertex for fermions is

$$g^{-1}\Lambda^{a}_{\psi\lambda}{}'(p) = T^a \gamma_\lambda Z_{1\psi} + \Gamma^a_\lambda(p) \qquad \ldots(4.31)$$

Thus the vertex renormalization constant $Z_{1\psi}$ is given by

$$Z_{1\psi} - 1 = \text{coefficient of } T^a\gamma_\lambda \text{ in } \Gamma^a_\lambda(p) \qquad \ldots(4.32)$$

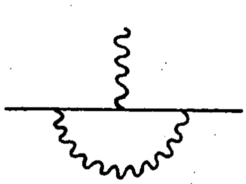
The diagrams contributing to the second order fermion vertex correction are listed in Figure 4.8. The general form of the vertex correction is:

$$\begin{aligned}\Gamma^a_\lambda(p) = -i\int d^{2\ell}k \{ & i\Lambda^b_{\psi\mu}\, i\Delta_\psi(p+k)\, i\Lambda^a_{\psi\lambda}\, i\Delta_\psi(p'+k)\times \\ & \times\, i\Delta^{bc,\mu\nu}(k)\, i\Lambda^c_{\psi\nu} \\ & + i\Lambda^b_{\psi\mu}\, i\Delta^{bb',\mu\sigma}(p-k)\, i\Lambda^{ab'c',\lambda\sigma\rho}(q,p-k,p'-k)\times \\ & \times\, i\Delta^{cc',\rho\nu}(p'-k)\, i\Delta_\psi(k)\, i\Lambda^c_{\psi\nu} \}\end{aligned} \qquad \ldots(4.33)$$

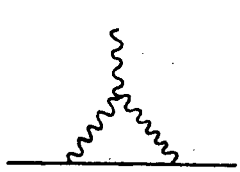
Specific formulae for $\Gamma^a_\lambda(p)|_\infty$ in the covariant, axial and Coulomb gauges for each diagram appear in (G47), (G49); (G87), (G90); (G154) and (G161) respectively.

The same notational procedure is followed as for the scalar vertex. Once again, calculations are done diagram by diagram in Appendix G, with special attention to detail in the Coulomb gauge. The major points are summarised here.

In *covariant* gauges, each diagram gives a separately covariant result. They are, from (G48) and (G50):



$$\Gamma^a_{\lambda(i)}(p)\Big|_\infty = -\gamma_\lambda T^a \frac{c}{2n} L$$

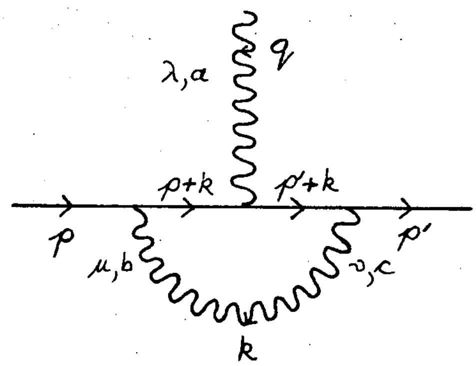


$$\Gamma^a_{\lambda(ii)}(p)\Big|_\infty = \gamma_\lambda T^a \frac{3(1+c)n}{4} L$$

Whence, from (G516):

$$\Gamma^a_\lambda(p)\Big|_\infty = \gamma_\lambda T^a \left[\frac{3(1+c)n^2 - 2c}{4n}\right] L \qquad \ldots(4.34c)$$

FERMION VERTEX CORRECTION



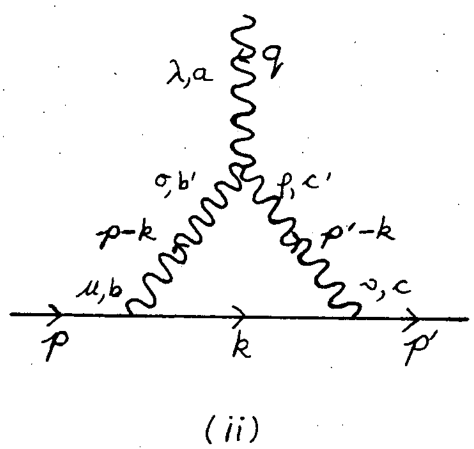
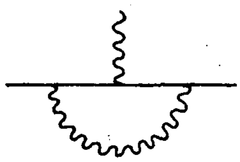
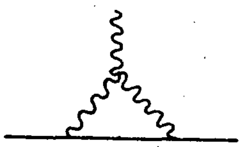


FIGURE 4.8 Diagrams contributing to the second order fermion vertex correction. They yield $Z_{1\psi}$.

In the *axial* gauge, each diagram is also separately covariant (contrast this with the scalar vertex). From (G88), (G89), (G91) and (G92) the separate terms adding to the vertex correction are:



$$\Gamma^a_{\lambda(i)}(p)\Big|_\infty = -\gamma_\lambda T^a \tfrac{3}{2n} L$$

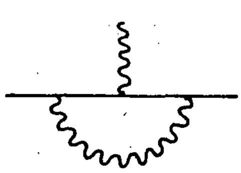


$$\Gamma^a_{\lambda(ii)}(p)\Big|_\infty = \gamma_\lambda T^a \tfrac{3n}{2} L$$

Whence, from (G95b):

$$\Gamma^a_{\lambda}(p)\Big|_\infty = \gamma_\lambda T^a \Big[\tfrac{3(1-n^2)}{2n} \Big] L \qquad \text{...(4.34a)}$$

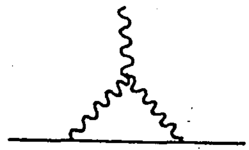
Once again, the *Coulomb* gauge presents the most interesting situation. Diagram (i) is covariant, whilst diagram (ii) is non-covariant. From (G157), (G160), (G165) and (G169) the terms adding to the total vertex correction are:



$$\Gamma^a_{o(i)}(p)\Big|_\infty = -\gamma_o T^a \tfrac{1}{2n} L$$

(virtual quantum transverse)

$$\Gamma^a_{\ell(i)}(p)\Big|_\infty = -\gamma_\ell T^a \tfrac{1}{2n} L$$



$$\Gamma^a_{o(ii)}(p)\Big|_\infty = \gamma_o T^a \tfrac{n}{2} L$$

(both virtual quanta transverse)

$$\Gamma^a_{\ell(ii)}(p)\Big|_\infty = \gamma_\ell T^a \tfrac{11n}{6} L$$

Whence, from (G170b) and (G171b):

$$\Gamma^a_{o}(p)\Big|_\infty = \gamma_o T^a \Big[\tfrac{n^2-1}{2n} \Big] L$$

$$\text{...(4.34r)}$$

$$\Gamma^a_{\ell}(p)\Big|_\infty = \gamma_\ell T^a \Big[\tfrac{11n^2-3}{6n} \Big] L$$

From the separate covariance of diagram (i) in the Coulomb gauge comes the following observation: If the gauge field were abelian, diagram (ii) would be absent. Therefore, in spinor electrodynamics, one could covariantly renormalize the fermion vertex, as has been noticed by Heckathorn [5]. However, this is not the case for the scalar vertex, and so Heckathorn's inductive proof of the covariance of pole parts in the Coulomb gauge for QED does not carry through into scalar electrodynamics.

Using (4.32) the renormalization constants are:

$$Z_{1\psi}\Big|_\infty = \begin{cases} 1 - \left[\dfrac{3(1+c)n^2 - 2c}{4n}\right] L & \ldots(4.35c) \\[2ex] 1 + \dfrac{3(n^2-1)}{2n} L & \ldots(4.35a) \\[2ex] \left.\begin{array}{ll} (Z_{1\psi}^{\parallel}) & 1 - \dfrac{n^2-1}{2n} L \\[2ex] (Z_{1\psi}^{\perp}) & 1 - \dfrac{11n^2-3}{6n} L \end{array}\right\} & \ldots(4.35r) \end{cases}$$

GHOST WAVE FUNCTION

The complete inverse propagator for the ghost field in covariant gauges is

$$\Delta_\chi^{aa'\,\prime\,-1}(p) = Z_{3\chi}\,\delta^{aa'}p^2 + \Pi^{aa'}(p) \qquad \ldots(4.36c)$$

whilst in the Coulomb gauge it takes the form

$$\Delta_\chi^{aa'\,\prime\,-1}(p) = -Z_{3\chi}\,\delta^{aa'}\not{p}^2 + \Pi^{aa'}(p) \qquad \ldots(4.36r)$$

In either case, the wave-function renormalization constant is given by

$$Z_{3\chi} - 1 = \text{coefficient of } \delta^{aa'}\not{p} \text{ in } \Pi^{aa'}(p) \qquad \ldots(4.37)$$

The diagrams involved in the second order ghost self-energy are to be found in Figure 4.9. The general form for $\Pi^{aa'}(p)$ is

$$\Pi^{aa'}(p) = -i\int d^{2\ell}k\, i\Lambda_{\chi\,\mu}^{acb}(p+k)\, i\Delta_\chi^{bb'}(p+k)\times$$
$$\times\, i\Delta^{aa';\mu\nu}(k)\, i\Lambda_{\chi\,\nu}^{a'b'c'}(p)$$
$$+ \text{ tadpole} \qquad \ldots(4.38)$$

GHOST SELF-ENERGY

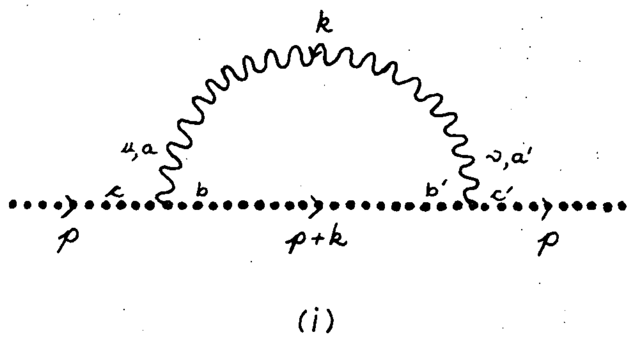


FIGURE 4.9 The diagram contributing to the second order ghost self-energy. It yields $Z_{3\chi}$.

GHOST VERTEX CORRECTION

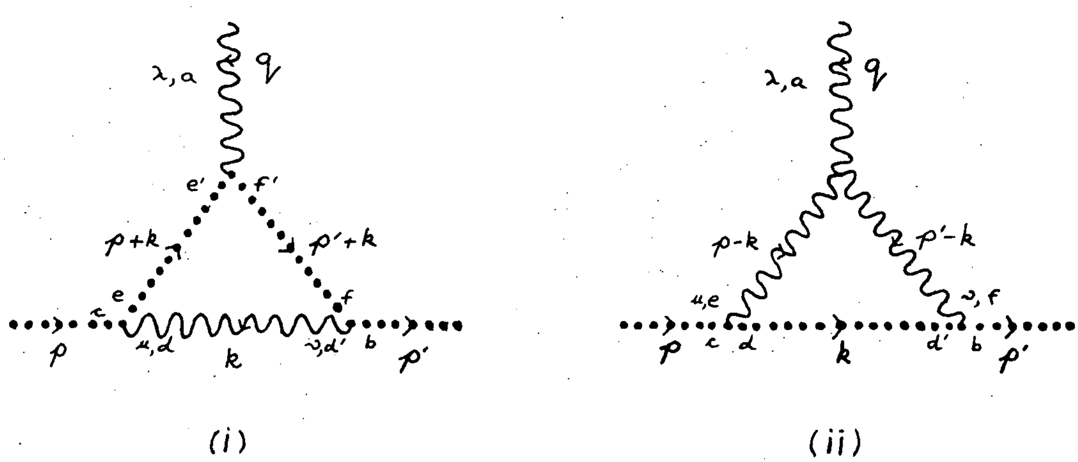


FIGURE 4.10 Diagrams contributing to the second order ghost vertex correction. They yield $Z_{1\chi}$.

The specific formulae pertaining to the covariant and Coulomb gauges appear in (G52) and (G172) respectively. It is interesting that in the Coulomb gauge, since the longitudinal vector mesons decouple from the ghost field, the self-energy here is purely due to the exchange of a transverse quantum. In summary, the three cases give, from (G53) and (G173):

$$\Pi^{aa'}(p)\Big|_{\infty} = \begin{cases} \delta^{aa'} p^2 \frac{c-3}{4} C_A L = \delta^{aa'} p^2 \frac{(c-3)n}{4} L & \ldots(4.39c) \\ 0 & \ldots(4.39a) \\ \delta^{aa'} p^2 \frac{4}{3} C_A L = \delta^{aa'} p^2 \frac{4n}{3} L & \ldots(4.39r) \end{cases}$$

yielding via (4.37) the wave-function renormalization constant

$$Z_{3\chi}\Big|_{\infty} = \begin{cases} 1 - \frac{c-3}{4} C_A L = 1 - \frac{(c-3)n}{4} L & \ldots(4.40c) \\ 1 & \ldots(4.40a) \\ 1 + \frac{4}{3} C_A L = 1 + \frac{4n}{3} L & \ldots(4.40r) \end{cases}$$

GHOST VERTEX

The proper scattering vertex for the ghost field in covariant gauges is

$$g^{-1} \Lambda'^{\,abc}_{\chi\lambda}(p) = i f^{abc} p_\lambda Z_{1\chi} + \Gamma^{abc}_{\lambda}(p) \qquad \ldots(4.41c)$$

and in the Coulomb gauge is

$$g^{-1} \Lambda'^{\,abc}_{\chi\lambda}(p) = i f^{abc}[p_\lambda - \eta_{\lambda 0} p_0] + \Gamma^{abc}_{\lambda}(p) \qquad \ldots(4.41v)$$

In either case, the vertex renormalization constant can be computed from

$$Z_{1\chi} - 1 = \text{coefficient of } i f^{abc} p_\ell \text{ in } \Gamma^{abc}_{\lambda}(p) \qquad \ldots(4.42)$$

The diagrams involved in the second-order ghost vertex correction are to be found in Figure 4.10. The general form for $\Gamma^{abc}_{\lambda}(p)$ is

$$\Gamma^{abc}_{\lambda}(p) = -i \int d^{2\omega}k \, i\Lambda^{ecd}_{\chi u}(p+k) \, i\Delta^{ee'}_{\chi}(p+k) \, i\Delta^{dd',u\nu}(k)$$
$$\times \, i\Lambda^{ae'f'}_{\chi\lambda}(p'+k) \, i\Delta^{f'f}_{\chi}(p'+k) \, i\Delta^{fd'b}_{\chi\nu}(p') \qquad \ldots(4.43)$$

Specific formulae for the covariant and Coulomb gauges appear in (G54), (G56); and (G174), (G177) respectively. The results are in (G58) and (G180).

To summarize:

$$\Gamma^{a\,bc}_{\lambda}(p)\Big|_{\infty} = \begin{cases} i f^{abc} p_\lambda \frac{c}{2} C_A L = i f^{abc} p_\lambda \frac{cn}{2} L & \text{...(4.44c)} \\ 0 & \text{...(4.44a)} \\ 0 & \text{...(4.44r)} \end{cases}$$

Thus, by (4.42) the vertex renormalization constants are

$$Z_{1\chi}\Big|_{\infty} = \begin{cases} 1 - \frac{cn}{2} L & \text{...(4.45c)} \\ 1 & \text{...(4.45a)} \\ 1 & \text{...(4.45r)} \end{cases}$$

4.3 ANALYSIS

Firstly, we seek to verify the Taylor identity [11] which states that Z_1/Z_3 is source independent (but gauge dependent). For the ghost field, (4.40) and (4.45) give

$$Z_{1\chi}/Z_{3\chi} = \begin{cases} 1 - \frac{(c+3)n}{4} L & \text{...(4.46c)} \\ 1 & \text{...(4.46a)} \\ 1 - \frac{4n}{3} L & \text{...(4.46r)} \end{cases}$$

In comparison, from (4.8) and (4.30), and (4.13) and (4.35), the scalar meson and fermion fields provide

$$Z_{1\phi}/Z_{3\phi} = Z_{1\psi}/Z_{3\psi} = \begin{cases} 1 - \frac{(c+3)n}{4} L & \text{...(4.47c)} \\ 1 & \text{...(4.47a)} \end{cases}$$

$$Z_{1\phi}/Z_{3\phi} = Z_{1\psi}/Z_{3\psi} = \left\{\begin{array}{ll} (Z_1^{\parallel}/Z_3) & 1 \\ & \\ (Z_1^{\perp}/Z_3) & 1 - \frac{4n}{3}L \end{array}\right\} \qquad \ldots(4.47r)$$

Immediately, the equality alluded to is apparent. Of course, it holds only for those 'physical' vertex renormalization constants involving the external transverse field.

Note the especially simple relation

$$Z_1^{\parallel}/Z_3 = 1 \qquad \ldots(4.48r)$$

for the longitudinal Coulomb field, which is decoupled from ghost interactions. This also holds in the ghost-free axial gauge for the total corrections, viz:

$$Z_1/Z_3 = 1 \qquad \ldots(4.48a)$$

Having confirmed the source independence of Z_1/Z_3, it is now possible to extrapolate from the Z_{3A} of (4.20) to find Z_{1A} using

$$\left.\begin{array}{rl} Z_{1A}^{\parallel} & = Z_{1A}^{\perp} \\ & = Z_{1A} \\ & = Z_{3A} Z_{1X}/Z_{3X} \end{array}\right\} \qquad \begin{array}{l}\ldots(4.49c) \\ \\ \ldots(4.49a)\end{array}$$

for the covariant and axial gauges, and

$$\left.\begin{array}{rl} Z_{1A}^{\parallel} & = Z_{3A}^{\parallel} \\ & \\ Z_{1A}^{\perp} & = Z_{3A}^{\perp} Z_{1X}/Z_{3X} \end{array}\right\} \qquad \ldots(4.49r)$$

for the Coulomb gauge.

Whence, for each gauge,

$$Z_{1A}\Big|_{\infty} = \left\{\begin{array}{lll} & 1 + \frac{(17-9c)n}{12} L & \ldots(4.50c) \\ & & \\ & 1 + \frac{11n}{3} L & \ldots(4.50a) \\ & & \\ (Z_{1A}^{\parallel}) & 1 + \frac{11n}{3} L & \\ & & \ldots(4.50r) \\ (Z_{1A}^{\perp}) & 1 - \frac{n}{3} L & \end{array}\right.$$

At this stage it is convenient to place all the renormalization constants together in Table 4.11. They have been checked against other results quoted in the literature, as follows:

Covariant gauges: Z_{3A}, Z_{1A} and $Z_{1\psi}$ [12,13]; $Z_{3\chi}$, $Z_{1\chi}$ and $Z_{3\psi}$ [13].

Axial gauge : Z_{3A} $(=Z_{1A})$ [3,4,14,15,16]; $Z_{3\phi}(=Z_{1\phi})$ [3]; $Z_{3\psi}(=Z_{1\psi})$ [15].

Coulomb gauge : $Z_{3A}^{\perp}$, $Z_{1A}^{\perp}$, $Z_{3\chi}$ and $Z_{1\chi}$ (all in $SU(2)$ only) [17]; $Z_{3A}^{\parallel}$ [4].

So Table 4.11 extends, and in some cases generalizes, the list of renormalization constants, as well as collecting them together in the one place.

Now that Z_{1A} has been evaluated in (4.50) it is possible to confirm the gauge invariance of equation (3.11), that is:

$$g_0/g = Z_{1A}/Z_{3A}^{3/2} \qquad \ldots(4.51)$$

From (4.20) and (4.50) we find

$$\begin{aligned}(Z_{1A}/Z_{3A}^{3/2})_{cov.} &= (Z_{1A}/Z_{3A}^{3/2})_{axial} \\ &= (Z_{1A}^{\parallel}/Z_{3A}^{\parallel\,3/2})_{coul.} = (Z_{1A}^{\perp}/Z_{3A}^{\perp\,3/2})_{coul.} \\ &= 1 - \tfrac{11n}{6} L \end{aligned} \qquad \ldots(4.52)$$

which is just the value quoted by others as leading to asymptotic freedom [4,12,13].

At the beginning of this chapter we mentioned certain covariant gauges which give equivalent renormalization constants to those in the axial or Coulomb gauges. The easiest way to explore coincidences of this type is to construct a list of 'equivalent κ-values,' i.e. those values of κ which, when used in the covariant gauge renormalization constants of Table 4.11, yield the same result as that given for the axial or Coulomb gauges. This is done in Table 4.12.

The most striking feature of Table 4.12 is the exact equivalence between the covariant $(\kappa=-3)$ gauge and the axial gauge, disregarding ghost sources. It is this correspondence

RENORMALIZATION CONSTANTS IN THE COVARIANT, AXIAL AND COULOMB GAUGES

Renorm. Constant	In covariant gauges (parameter c)	In the axial gauge	In the Coulomb gauge
$Z_{3\phi}-1$	$\frac{(3-c)(n^2-1)}{2n}L$	$\frac{3(n^2-1)}{n}L$	$\frac{n^2-1}{n}L$
$Z_{3\psi}-1$	$\frac{(1-n^2)c}{2n}L$	$\frac{3(n^2-1)}{2n}L$	$\frac{1-n^2}{2n}L$
$Z_{3A}^{\parallel}-1$	$\frac{(13-3c)n}{6}L$	$\frac{11n}{3}L$	$\frac{11n}{3}L$
$Z_{3A}^{\perp}-1$	same	same	nL
$Z_{3\chi}-1$	$\frac{(3-c)n}{4}L$	0	$\frac{4n}{3}L$
$Z_{1\phi}^{\parallel}-1$	$\frac{3(1-c)n^2+2c-6}{4n}L$	$\frac{3(n^2-1)}{n}L$	$\frac{n^2-1}{n}L$
$Z_{1\phi}^{\perp}-1$			$\frac{-(n^2+3)}{3n}L$
$Z_{1\psi}^{\parallel}-1$	$\frac{-3(1+c)n^2+2c}{4n}L$	$\frac{3(n^2-1)}{2n}L$	$\frac{1-n^2}{2n}L$
$Z_{1\psi}^{\perp}-1$	same	same	$\frac{3-11n^2}{6n}L$
$Z_{1A}^{\parallel}-1$	$\frac{(17-9c)n}{12}L$	$\frac{11n}{3}L$	$\frac{11n}{3}L$
$Z_{1A}^{\perp}-1$	same	same	$-\frac{n}{3}L$
$Z_{1\chi}-1$	$\frac{-cn}{2}L$	0	0

TABLE 4.11 Wave function and vertex renormalization constants for scalar ϕ, spinor ψ, vector A and fictitious χ fields in the covariant, axial and Coulomb gauges. L is the divergent constant $\frac{g^2}{16\pi^2}\ln\frac{\Lambda^2}{\mu^2}$. The underlying group is $SU(n)$.

EQUIVALENT c-VALUES

Renorm. Constant	axial gauge equiv. c-value	Coulomb gauge equiv. c-value
$Z_{3\phi}, Z_{3\psi}$	-3	1
$Z_{1\phi}^{\parallel}, Z_{1\psi}^{\parallel}$	-3	$-\frac{n^2+2}{3n^2-2}$
$Z_{1\phi}^{\perp}, Z_{1\psi}^{\perp}$	-3	$\frac{13n^2-6}{9n^2-6}$
$Z_{3A}^{\parallel}, Z_{1A}^{\parallel}$	-3	-3
$Z_{3A}^{\perp}, Z_{1A}^{\perp}$	-3	$\frac{7}{3}$
$Z_{3\chi}$	$+3$	$-\frac{7}{3}$
$Z_{1\chi}$	0	0

TABLE 4.12 Values of the parameter c which, when used in the covariant gauges, yield the same wave-function or vertex renormalization constants as those found in the non-covariant axial and Coulomb gauges.

which has been reported in references [1,2,3]. The only clue to this phenomenon is that in the covariant gauge with parameter $\kappa=-3$ we find

$$Z_1 / Z_3 = 1 \qquad \ldots(4.53)$$

exactly, in analogy with (4.48a) for the axial gauge. (4.53) cannot imply any decoupling mechanism for the ghost field in this covariant gauge - hence the different values of κ required for the ghost wave-function and vertex in Table 4.12.

For the Coulomb gauge the situation is more subtle. Certainly, scalar meson and fermion wave-function renormalizations are the same here as they are in the Fermi $\kappa=1$ gauge. However, vector-meson and ghost renormalizations separately require different values of κ. And for scalar meson and fermion vertex renormalizations the equivalent κ-values are dependent on the number of group generators, n.

One might pin the blame for the discrepancy in the vector meson case on the presence of ghosts in the Coulomb gauge. But the ghost field takes no part in the calculations of $\Pi_{00}(p)|_\infty$ or $\Pi_{kk}(p)|_\infty$ resulting in (G117) and (G123). So the mechanism for this ghost interference must be more subtle than a simple 'additional diagram' type approach.

In the next chapter the above-mentioned correspondences will be explored further by repeating the calculations in the general gauge of Section 3.6.

* * * * * * * * * * *

REFERENCES - CHAPTER 4

1. T.P. Beven and R. Delbourgo : Lett.Nuovo Cim. 23, 433 (1978).
2. T. Curtright and G. Ghandour : Ann. Phys. 112, 237 (1978).
3. R. Delbourgo, A. Salam and J. Strathdee : Nuovo Cim. A23, 237 (1974).

4. J. Frenkel and J.C. Taylor : Nucl. Phys. B109, 439 (1976).

5. D. Heckathorn : Nucl. Phys. B156, 328 (1979).

6. G. Leibbrandt : Rev. Mod. Phys. 47, 849 (1975).

7. W. Kummer : Acta Phys. Austriaca 41, 315 (1975).

8. W. Konetschny and W. Kummer : Nucl. Phys. B108, 397 (1975).

9. P. Cvitanovic : Nucl. Phys. B130, 114 (1977).

10. W. Konetschny : Nuovo Cim. A44, 465 (1978).

11. J.C. Taylor : Nucl. Phys. B33, 436 (1971).

12. H.D. Politzer : Phys. Rev. Lett. 30, 1346 (1973).

13. D.J. Gross and F. Wilczek : Phys. Rev. D8, 3633 (1973).

14. W. Kainz, W. Kummer and M. Schweda : Nucl. Phys. B79, 484 (1974).

15. J. Frenkel and R. Meuldermans : Phys. Lett. B65, 64 (1974).

16. J. Frenkel : Phys. Lett. B60, 74 (1975).

17. A. Ali, J. Bernstein and A. Zepeda : Phys. Rev. D12, 503 (1975).

5. RENORMALIZATION CONSTANTS IN THE GENERAL GAUGE

Following the presentation of second order renormalization constants for the Lorentz covariant (parameter α), axial and Coulomb gauges in Chapter 4, we now repeat the analysis in the general gauge of Section 3.6. This chapter is by nature short, it being quite unnecessary to repeat the figures or formulae relating to each second order self-energy or vertex correction. But this should not be taken to infer that what follows is of lesser importance than preceeding results. Indeed, the generality of the approach used here is reflected both in the complexity of the computations involved and in the broader applicability of the results obtained.

5.1 PREAMBLE

Frenkel and Taylor [1] first introduced a general gauge fixing term which incorporates the Lorentz covariant, axial and Coulomb gauges, in the course of a discussion concerning the basis of asymptotic freedom. For simplicity, they restricted themselves to the usual axial and Coulomb gauge limits.

This concept has been further developed in Section 3.6 of this thesis through the derivation of the Yang-Mills propagator and the associated fictitious scalar propagator and interaction vertex in this general gauge. (see also Appendix C in this regard). In this chapter, the Feynman rules of Table 3.4 are used to calculate the second order renormalization constants, and then various limits of the arbitrary parameters are taken so as to study correspondences between different particular gauge choices. These results were first reported in reference [2].

To recapitulate from Section 3.6, the gauge fixing term is (3.63):

$$\mathcal{L}_{g.f.} = -\frac{1}{2}\left[(b^2-a^2)n^\mu A_\mu \frac{\partial.n}{n^2} + a^2\partial^\mu A_\mu\right]^2 \qquad \ldots(5.1)$$

where a and b are arbitrary constants which yield, with various limits, the gauges:

Covariant gauges : $a^2 = b^2 \equiv \alpha^{-1/4}$.

Axial gauge : $a^2 = 0$, $b^2 \to \infty$.

Coulomb gauge : $b^2 = 0$, $a^2 \to \infty$.

When (5.1) is introduced into the functional formalism it produces the vector gauge meson propagator of (3.68), viz:

$$\Delta^{ab}_{\mu\nu}(k) = \frac{\delta^{ab}}{k^2}\left[-\eta_{\mu\nu} + \frac{k_\mu K_\nu + k_\nu K_\mu}{k.K} - \frac{k_\mu k_\nu K^2}{k.K}\left(1+\frac{k^2}{K^2}\right)\right] \qquad \ldots(5.2)$$

where

$$K_\mu = a^2 k_\mu + (b^2-a^2)\frac{k.n}{n^2}n_\mu \qquad \ldots(5.3)$$

In addition, as a consequence of the Faddeev-Popov procedure [3], there is the fictitious ghost field propagator

$$\Delta^{ab}_\chi(k) = \frac{\delta^{ab}a^2}{k.K} \qquad \ldots(5.4)$$

and a one-sided ghost-vector meson vertex

$$\Lambda^{abc}_{\chi\,\mu}(p',p) = i g f^{abc}\frac{1}{a^2}P_\mu \qquad \ldots(5.5)$$

where

$$P_\mu = a^2 p_\mu + (b^2-a^2)p.n\, n_\mu \qquad \ldots(5.6)$$

The comments in Section 4.1 relating to calculational procedure apply here also, except for some minor clarifications relating to the broadness of this approach.

The first point has to do with the singularities of the propagator (5.2) in the $k.n$ plane. To be more specific, when n_μ is a unit timelike vector, the propagator becomes (C16):

$$\Delta^{ab}_{\mu\nu}(k) = \frac{\delta^{ab}}{k^2}\left[-\eta_{\mu\nu} + \frac{(b^2-a^2)k_0(k_\mu\eta_{\nu 0}+k_\nu\eta_{\mu 0})}{(b^2k_0^2-a^2\underline{k}^2)} - \frac{(b^2-a^2)^2k_0^2+(1-a^4)k^2}{(b^2k_0^2-a^2\underline{k}^2)^2}k_\mu k_\nu\right] \qquad \ldots(5.7)$$

The singularities in the k_0 plane are now located at

$$k_0 = \pm\frac{a}{b}|\underline{k}| \quad \text{and} \quad k_0 = \pm|\underline{k}|$$

with a and b taken positive. When $a^2=b^2\equiv\alpha^{-1/4}$ these singularities will coalesce and it is appropriate to interpret the resulting covariant propagator by a $k^2+i\epsilon$ prescription for single and double poles. But when $a^2 \neq b^2$, (5.7) is non-covariant and it becomes important to adopt a principal-value prescription for the zero of $b^2k_0^2 - a^2\underline{k}^2$ in order to preserve on-shell unitarity. A simple and practical way to proceed is to interpret the denominator in (5.7) as $b^2k_0^2 - a^2\underline{k}^2 + i\epsilon$ and later, for non-covariant gauges where $a,b \neq 0$, to average out over the signs of a and b in order to take principal values of integrals (see Appendix D in this regard). The fundamental divergent integrals involving a and b are labelled J_1 etc., and worked out in Appendix D. They are listed in Table D2.

Another point worth mentioning is that in this chapter, in contrast to Chapter 4, the representations for the scalar meson and fermion fields are kept quite general. This has to do with the fact that here the only vertex renormalization constant explicitly evaluated is Z_{1X}, the others being obtained via the Taylor identities [4]. Armed with the knowledge gained in Chapter 4 concerning the covariance of kinematic factors for the covariant and axial gauges, and understanding the peculiarities of the Coulomb gauge, we are quite happy to express the results in Table 5.1 in terms of combinations of the quadratic Casimir operators in each representation.

When necessary, reference is made to Chapter 4 for diagrams and applicable equations, and to Appendix H for calculational results. The cases covered there are the scalar meson, fermion and ghost self-energies, and the ghost vertex correction. We also quote the result [2] for the vector meson self-energy. From these, a complete list of renormalization constants can be drawn up.

For completeness, both the covariant (c) and general non-covariant (n) limits are listed here, even though the former have appeared previously.

The propagator (5.7) can be written in a matrix form as

$$\Delta_{\mu\nu}(k) = \begin{bmatrix} \frac{a^4 \underline{k}^2 - k_0^2}{(b^2 k_0^2 - a^2 \underline{k}^2)^2} & \frac{a^2 b^2 - 1}{(b^2 k_0^2 - a^2 \underline{k}^2)^2} k_0 k_j \\ \frac{a^2 b^2 - 1}{(b^2 k_0^2 - a^2 \underline{k}^2)^2} k_i k_0 & \frac{1}{\underline{k}^2}\left(\delta_{ij} - \frac{k_i k_j}{\underline{k}^2}\right) + \frac{b^4 k_0^2 - \underline{k}^2}{(b^2 k_0^2 - a^2 \underline{k}^2)^2} \frac{k_i k_j}{\underline{k}^2} \end{bmatrix} \quad \ldots (5.8)$$

in analogy with (4.1).

5.2 EVALUATION OF RENORMALIZATION CONSTANTS

SCALAR MESON WAVE-FUNCTION

Figure 4.2 provides the relevant diagrams. The formula for the scalar meson self-energy in the general gauge appears in (H1). From (H3), the results are:

$$\Pi(p^2)\Big|_\infty = \begin{cases} [-3p^2 + c(p^2 - m^2)]\, C_\phi L & \ldots (5.9c) \\ [-2p^2 - m^2 - (p^2 - m^2)\frac{4b^2}{b^2 - a^2}]\, C_\phi L & \ldots (5.9n) \end{cases}$$

Hence, using (4.5), the wave-function renormalization constants are:

$$Z_{3\phi}\Big|_\infty = \begin{cases} 1 + (3 - c)\, C_\phi L & \ldots (5.10c) \\ 1 + \frac{2a^2 - 6b^2}{a^2 - b^2}\, C_\phi L & \ldots (5.10n) \end{cases}$$

FERMION WAVE-FUNCTION

Figure 4.3 provides the relevant diagrams. The formula for the fermion self-energy in the general gauge appears in (H4). From (H6), the results are

$$\Sigma(p)\Big|_\infty = \begin{cases} [-3m + c(\not{p} - m)]\, C_\psi L & \ldots (5.11c) \\ [\not{p} - 4m + (\not{p} - m)\frac{4b^2}{b^2 - a^2}]\, C_\psi L & \ldots (5.11n) \end{cases}$$

where we immediately note the similarity between (5.11n) and (5.9n). Hence, using (4.10), the wave-function renormalization constants are:

$$Z_{3\psi}\Big|_{\infty} = \begin{cases} 1 - c\, C_\psi L & \text{...(5.12c)} \\ 1 - \dfrac{a^2+3b^2}{a^2-b^2}\, C_\psi L & \text{...(5.12n)} \end{cases}$$

VECTOR MESON WAVE-FUNCTION

Figure 4.4 gives the pure Yang-Mills and ghost diagrams which contribute to the self-energy calculation. As detailed in Section 4.2, the doubly transverse nature of the vector meson self-energy necessitates the decomposition in (4.16), namely

$$\Pi_{KK'}(p) = (-\eta_{KK'}p^2 + p_K p_{K'})\Pi_C + (p^2 n_K - p.n\, p_K)(p^2 n_{K'} - p.n\, p_{K'})\Pi_N + [(p^2 n_K - p.n\, p_K)n_{K'} + (p^2 n_{K'} - p.n\, p_{K'})n_K]\,\Pi_M \quad \text{...(5.13)}$$

where Π_C , Π_N and Π_M are scalar functions of p^2 and $p.n$. Π_N is finite, and the results for the other two coefficients are [2]:

$$\Pi_C = \begin{cases} \dfrac{3c-13}{6}\, C_A L & \text{...(5.14c)} \\ -\dfrac{3a^2+11b^2}{3(a^2-b^2)}\, C_A L & \text{...(5.14n)} \end{cases}$$

and

$$\Pi_M = \begin{cases} 0 & \text{...(5.15c)} \\ \dfrac{4a^2}{3(a^2-b^2)}\, C_A L & \text{...(5.15n)} \end{cases}$$

Hence, using (4.19), the wave-function renormalization constants are:

$$Z_{3A}^{\parallel}\Big|_{\infty} = \begin{cases} 1 + \dfrac{13-3c}{6}\, C_A L & \text{...(5.16c)} \\ 1 + \dfrac{11}{3}\, C_A L & \text{...(5.16n)} \end{cases}$$

and

$$Z_{3A}^{\perp}\Big|_{\infty} = \begin{cases} 1 + \dfrac{13-3c}{6}\, C_A L & \text{...(5.17c)} \\ 1 + \dfrac{3a^2-11b^2}{3(a^2-b^2)}\, C_A L & \text{...(5.17n)} \end{cases}$$

GHOST WAVE-FUNCTION

Figure 4.9 provides the relevant diagrams. The formula for the ghost self-energy in the general gauge appears in (H8). From (H9) the results are:

$$\Pi^{aa'}(p)\Big|_\infty = \begin{cases} \delta^{aa'} p^2 \, \frac{c-3}{4} \, C_A L & \text{...(5.18c)} \\ \delta^{aa'} p^2 \, \frac{4a^2}{3(a^2-b^2)} \, C_A L & \text{...(5.18n)} \end{cases}$$

Hence, using (4.37), the wave function renormalization constants are:

$$Z_{3x}\Big|_\infty = \begin{cases} 1 + \frac{3-c}{4} C_A L & \text{...(5.19c)} \\ 1 + \frac{4a^2}{3(a^2-b^2)} C_A L & \text{...(5.19n)} \end{cases}$$

GHOST VERTEX

Figure 4.10 shows the two diagrams contributing to the ghost vertex correction. In the general gauge, the formulae for diagrams (i) and (ii) appear in (H10) and (H13) respectively. It is interesting to observe from (H11) and (H12) and (H17) and (H18), that each diagram's divergent part <u>separately</u> involves the kinematic factor $[p_\lambda + \frac{b^2-a^2}{a^2} p_0 \eta_{\lambda 0}]$, which is just the bare vertex (up to factors of i and g). $\Gamma_{\lambda(i)}(p)\big|_\infty$ and $\Gamma_{\lambda(ii)}(p)\big|_\infty$ provide $1/2$ and $3/4$ respectively of the total divergent part.

From (H19), the result prior to the application of the covariant or non-covariant limits is:

$$\Gamma^{abc}_{\lambda}(p)\Big|_\infty = i\,[p_\lambda + \frac{(b^2-a^2)}{a^2} p_0 \eta_{\lambda 0}]\,(\tfrac{1}{2}\tfrac{1}{a^3 b}) C_A f^{abc} L \quad \text{...(5.20)}$$

After either putting $a = b \equiv c^{-1/4}$ or averaging over the sign of a, we arrive at

$$\Gamma^{abc}_{\lambda}(p)\Big|_\infty = \begin{cases} i f^{abc} p_\lambda \frac{c}{2} C_A L & \text{...(5.21c)} \\ 0 & \text{...(5.21n)} \end{cases}$$

Hence, from (4.42), the vertex renormalization constants are:

$$Z_{1x}\Big|_{\infty} = \begin{cases} 1 - \frac{c}{2} C_A L & \quad \ldots(5.22c) \\ \\ 1 & \quad \ldots(5.22n) \end{cases}$$

5.3 ANALYSIS

Firstly, it is clear from section 5.2 that the process of working out the self-energies and vertex corrections in the general gauge and <u>then</u> passing to the covariant gauge limit reproduces the results previously found for this gauge in Chapter 4. Secondly, if the appropriate limits are applied to the non-covariant results of Section 5.2, one finds equality with the axial and Coulomb gauge calculations of Chapter 4. Moreover, the Taylor identities which were explicitly confirmed in Section 4.3 can now be used to find the remaining renormalization constants. From (5.19) and (5.22) we have:

$$Z_{1x}/Z_{3x} = \begin{cases} 1 - \frac{c+3}{4} C_A L & \quad \ldots(5.23c) \\ \\ 1 - \frac{4a^2}{3(a^2-b^2)} C_A L & \quad \ldots(5.23n) \end{cases}$$

Using (5.10), (5.12), (5.16), (5.17), (5.23) and the Taylor identity (3.110), as well as the knowledge that

$$Z''_{1A} = Z''_{3A}$$

in both the axial and Coulomb gauge limits of the non-covariant gauge, we deduce that

$$Z''_{1A}\Big|_{\infty} = \begin{cases} 1 + \frac{17-9c}{12} C_A L & \quad \ldots(5.24c) \\ \\ 1 + \frac{11}{3} C_A L & \quad \ldots(5.24n) \end{cases}$$

$$Z^{\perp}_{1A}\Big|_{\infty} = \begin{cases} 1 + \frac{17-9c}{12} C_A L & \quad \ldots(5.25c) \\ \\ \quad - \frac{a^2+11b^2}{3(a^2-b^2)} C_A L & \quad \ldots(5.25n) \end{cases}$$

$$Z_{1\phi}^{\perp}\Big|_{\infty} = \begin{cases} 1 - \frac{1}{4}\left[(3+\alpha)C_A + (4\alpha - 12)C_\phi\right]L & \text{...(5.26c)} \\[2ex] 1 - \dfrac{4a^2C_A - 6(a^2-3b^2)C_\phi}{3(a^2-b^2)}L & \text{...(5.26n)} \end{cases}$$

$$Z_{1\psi}^{\perp}\Big|_{\infty} = \begin{cases} 1 - \frac{1}{4}\left[(3+\alpha)C_A + 4\alpha C_\psi\right]L & \text{...(5.27c)} \\[2ex] 1 - \dfrac{4a^2C_A + 3(a^2+3b^2)C_\psi}{3(a^2-b^2)}L & \text{...(5.27n)} \end{cases}$$

Unfortunately, it is not possible to extrapolate to $Z_{1\phi}^{\parallel}$ or $Z_{1\psi}^{\parallel}$ since, as we saw in (4.46) and (4.47), the Taylor identity tells us nothing about these. These results are summarized in Table 5.1. In analogy with Table 4.12, we construct a list of 'equivalent α-values' - that is, those values of α which when used in the covariant gauge will yield the same value for the renormalization constant as that found for the non-covariant gauge. It is possible to reverse this view and consider those values of the parameter a^2/b^2 which, when used in non-covariant gauges, will reproduce the answers obtained in a covariant gauge with parameter α. This is all done in Table 5.2.

From this table one can see the equivalences mentioned in Section 4.3 between the covariant $(\alpha = -3)$ and axial gauges and between the Fermi $(\alpha = 1)$ and Coulomb gauges - the latter only for scalar meson and fermion wave-function Z's. There is also now the same connection between the Laudau $(\alpha = 0)$ gauge, and a non-covariant gauge with $a^2/b^2 = -3$. And of course an infinity of other such connections if one is to consider non-integer values of α. Regardless of which equivalent α-values (or a^2/b^2-values) are found for $Z_{3\phi}$ and $Z_{3\psi}$, there will always be a strict equality between the corresponding values found for $Z_{3A}^{\parallel}$ and $Z_{1A}^{\parallel}$, $Z_{3A}^{\perp}$ and $Z_{1A}^{\perp}$, and $Z_{1\phi}^{\perp}$ and $Z_{1\psi}^{\perp}$. This follows either from the ghost-free nature of the corrections,

RENORMALIZATION CONSTANTS IN THE COVARIANT AND GENERAL NON-COVARIANT GAUGES

Renorm. Constant	In covariant gauges (parameter c)	In non-covariant gauges (parameter a^2/b^2)
$Z_{3\phi}-1$	$(3-c)C_\phi L$	$\frac{2a^2-6b^2}{a^2-b^2} C_\phi L$
$Z_{3\psi}-1$	$-c\, C_\psi L$	$-\frac{a^2+3b^2}{a^2-b^2} C_\psi L$
$Z_{3A}^{\parallel}-1$	$\frac{13-3c}{6} C_A L$	$\frac{11}{3} C_A L$
$Z_{3A}^{\perp}-1$	same	$\frac{3a^2-11b^2}{3(a^2-b^2)} C_A L$
$Z_{3\chi}-1$	$\frac{3-c}{4} C_A L$	$\frac{4a^2}{3(a^2-b^2)} C_A L$
$Z_{1\phi}^{\perp}-1$	$-\frac{(3+c)C_A+4(c-3)C_\phi}{4} L$	$-\frac{4a^2C_A-6(a^2-3b^2)C_\phi}{3(a^2-b^2)} L$
$Z_{1\psi}^{\perp}-1$	$-\frac{(3+c)C_A+4c\,C_\psi}{4} L$	$-\frac{4a^2C_A+3(a^2+3b^2)C_\psi}{3(a^2-b^2)} L$
$Z_{1A}^{\parallel}-1$	$\frac{17-9c}{12} C_A L$	$\frac{11}{3} C_A L$
$Z_{1A}^{\perp}-1$	same	$-\frac{a^2+11b^2}{3(a^2-b^2)} C_A L$
$Z_{1\chi}-1$	$-\frac{1}{2}c\, C_A L$	0

TABLE 5.1 Wave function and vertex renormalization constants for scalar ϕ, spinor ψ, vector A and fictitious χ fields in the covariant and general non-covariant gauges. L is the divergent constant $\frac{g^2}{16\pi^2} \ln \frac{\Lambda^2}{\mu^2}$. The underlying group is $SU(n)$. Group representations are arbitrary for scalar and spinor fields. C_X is the quadratic Casimir operator in the representation chosen for the field X.

EQUIVALENT c -VALUES AND a^2/b^2 -VALUES

Renorm. Constant	Non-covariant gauge equiv. c-value	Covariant gauge equiv. a^2/b^2 -value
$Z_{3\phi}, Z_{3\psi}$	$\frac{a^2+3b^2}{a^2-b^2}$	$\frac{c+3}{c-1}$
$Z^{\perp}_{1\phi}$	$\frac{(7a^2+9b^2)C_A+12(a^2+3b^2)C_\phi}{3(a^2-b^2)C_A+12(a^2-b^2)C_\phi}$	$\frac{(3c+9)C_A+12(c+3)C_\phi}{(3c-7)C_A+12(c-1)C_\phi}$
$Z^{\perp}_{1\psi}$	$\frac{(7a^2+9b^2)C_A+12(a^2+3b^2)C_\psi}{3(a^2-b^2)C_A+12(a^2-b^2)C_\psi}$	$\frac{(3c+9)C_A+12(c+3)C_\psi}{(3c-7)C_A+12(c-1)C_\psi}$
$Z^{\parallel}_{3A}, Z^{\parallel}_{1A}$	-3	
$Z^{\perp}_{3A}, Z^{\perp}_{1A}$	$\frac{7a^2+9b^2}{3(a^2-b^2)}$	$\frac{3c+9}{3c-7}$
$Z_{3\chi}$	$-\frac{7a^2+9b^2}{3(a^2-b^2)}$	$-\frac{3c+9}{3c-7}$
$Z_{1\chi}$	0	
	(1)	(2)

TABLE 5.2 (1) Values of the parameter c which, when used in the covariant gauges, yield the same renormalization constants as those found in a non-covariant gauge characterized by the parameters a^2 and b^2.

(2) Values of the ratio a^2/b^2 which, when used in the general non-covariant gauge, yield the same renormalization constants as those found in a covariant gauge characterized by parameter c.

or from the Taylor identity, depending on whether the parallel or transverse part is considered.

It must be emphasised that these equivalences are proven here to one-loop level only. It is problematical as to whether they persist in higher orders of perturbation theory [5]. One expects, as in electrodynamics [6], that the κ will be modified by higher order corrections; that is, $\kappa = -3 + O(g^2)$ in the covariant gauge equivalent of the axial gauge, and likewise for the other equivalences.

* * * * * * * * *

REFERENCES - CHAPTER 5

1. J. Frenkel and J.C. Taylor : Nucl. Phys. B109, 439 (1976).

2. T.P. Beven and R. Delbourgo : Lett. Nuovo Cim. 27, 565 (1980); and erratum, 28, 400 (1980).

3. L.D. Faddeev and V.N. Popov : Phys. Lett. B25, 29 (1967)

4. J.C. Taylor : Nucl. Phys. B33, 436 (1971).

5. W. Konetschny : Nucl. Phys. B152, 153 (1979).

6. M. Baker, K.W. Johnson and J. Willey : Phys. Rev. 163, 1699 (1967).

6. CONCLUSION

Here we summarize the work presented in the thesis, and discuss the outlook for future related studies.

6.1 SUMMARY

In Chapter 2 we gave a review of the canonical quantization procedure for the pure electromagnetic field. We concentrated on the Lorentz, Coulomb, temporal and axial gauges. The difficulties associated with quantizing in each of these gauges (lack of consistency with Maxwell's equations, non-physical degrees of freedom, and the like) and the resolution thereof were discussed.

From there we moved on in Chapter 3 to outline the salient features of path-integral quantization of the Yang-Mills field. We gave the Feynman rules in the above gauges, as well as those in a general gauge which reduces to the others in various limits. Regularization and renormalization, and identities due to the gauge invariance, were also dealt with.

In Chapter 4 calculations of the various Z commenced. Details of the one-loop calculations were provided for the covariant, axial and Coulomb gauges. Transversality of the vector meson self-energy (as predicted by the Slavnov-Taylor identities) and similar features were verified to this order in each gauge. We also explicitly confirmed the source-independent Taylor identity (which states that Z_1/Z_3 is a constant for each gauge), as well as the gauge independence of the quantity $g_0/g = Z_{1A}Z_{3A}^{-3/2}$.

In the course of this investigation an exact equivalence was revealed between the Z's calculated using the axial gauge, and those found in the covariant gauge with $\alpha=-3$. Also uncovered was a parallelism between the Coulomb and Fermi ($\alpha=1$) gauges, particularly transparent for scalar and spinor wave-function renormalizations. Tables 4.11 and 4.12 summarized these results.

Using the general gauge described in Chapter 3, we moved on to repeat these one-loop calculations in Chapter 5. This gauge has two primary limits : the ordinary covariant gauges, and a general non-covariant gauge described by a parameter a^2/b^2 . The results of this investigation were collected together in Tables 5.1 and 5.2. Besides the equivalences mentioned above, we were able to find an equivalence between any of the covariant gauges and a non-covariant counterpart. In particular, the Landau ($\alpha=0$) gauge is equivalent to the non-covariant gauge with $a^2/b^2=-3$. However, the axial and ($\alpha=-3$) covariant gauge correspondence remains the most striking since it is exact for all counter terms.

6.2 OUTLOOK

The progress so far has been at the cost of tedious and involved manipulations - and these only for the second order calculations. It is thus imperative that any future work proceed with computer assistance. With this in mind, an investigation of the suitability of REDUCE [1] as an aid in problems of this type is warranted. It would be especially useful to carry out fourth order calculations in the general gauge using REDUCE - thus obtaining the results in each distinct gauge as a bonus. The second order results presented here would serve as a check. The nature of the equivalences discovered in this thesis might be found in higher orders. At the least one could obtain confirmation or otherwise of the claims of Hagen and Singh [2] concerning non-covariance of S-matrix elements.

Renormalization constants in a ghost-free gauge besides the axial gauge [3] would present a welcome addition to those listed here. Perhaps such a gauge will also have an exact covariant counterpart. If so, recalling that $Z_1/Z_3=1$ for the ($\alpha=-3$) covariant and axial gauges, perhaps this and other ghost-free gauges will be equivalent to these two.

Looking further afield, Matsuki [4,5] has commenced research into the axial gauge in gravity and supergravity. Capper and Delbourgo have also made progress in the study of gravity in axial gauges [6]. It may be that there is a corresponding 'covariant-type' gauge to the axial gauge in this theory as well. In the course of such work computer assistance would be vital, as has already been found by those involved in gravitational one-loop calculations (e.g. refs. [7,8]).

* * * * * * * * * *

REFERENCES - CHAPTER 6

1. A.C. Hearn : 'REDUCE 2 User's Manual,' Utah preprint, U C P-19 (1973).

2. C.R. Hagen and L.P.S. Singh : Phys. Rev. D21, 1620 (1980).

3. W. Konetschny : Phys. Lett. B90, 263 (1980).

4. T. Matsuki : Phys. Rev. D19, 2879 (1979) and erratum, D20, 2100 (1979).

5. T. Matsuki : KEK preprint, 79-16 (1979).

6. R. Delbourgo : private communication (Dec. 1980).

7. D.M. Capper and A. Namazie : Nucl. Phys. B142, 535 (1978).

8. D.M. Capper : Queen Mary College preprint QMC-79-11 (1979).

APPENDICES

APPENDIX A. NOTATION

The system of units used throughout is one in which $\hbar = c = 1$, where the 'bar' denotes division by 2π.

Four-vectors are denoted by k, p, A, B etc.

Three-vectors are denoted by $\underline{k}, \underline{p}, \underline{A}, \underline{B}$ etc.

Greek indices range over the values 0,1,2,3 and Latin indices over 1,2,3 except where otherwise stated.

The metric used is $\eta_{\mu\nu} = (+,-,-,-)$ with $\eta^{\mu}{}_{\nu} \equiv \delta_{\mu\nu}$, so that $x^2 = x_0^2 - \underline{x}^2$, where $\underline{x}^2 = x_1^2 + x_2^2 + x_3^2$.

The Dirac matrices obey the anticommutation rule

$$\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2\eta^{\mu\nu}$$

A slash through a four-vector indicates its inner product with a Dirac matrix, i.e. $\not{p} = \gamma^{\mu} p_{\mu}$.

Co-ordinate derivatives are most often indicated by $\partial^{\mu} = \frac{\partial}{\partial x_{\mu}}$.

$$\nabla \equiv \underline{\partial} = \left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3} \right)$$

$$\Box \equiv \partial^{\mu}\partial_{\mu} = \partial_0^2 - \underline{\partial}^2 = \left(\frac{\partial}{\partial x^0}\right)^2 - \left(\frac{\partial}{\partial x^1}\right)^2 - \left(\frac{\partial}{\partial x^2}\right)^2 - \left(\frac{\partial}{\partial x^3}\right)^2$$

Sometimes a dot is placed above a symbol to indicate its time derivative.

Other symbols are defined where they are used for the first time, or are in such common usage as to be self-explanatory.

There are two notable cases where the same symbol is used for different quantities. These are:

Z (renormalization constant, and vacuum generating functional), and

n (non-covariant gauge axis, and the order of the matrix representatives of the underlying gauge group $SU(n)$).

It should also be noted that the symbol m is employed for both the scalar meson and fermion masses in Chapters 4 and 5 and Appendices G and H. No confusion should arise from this usage.

APPENDIX B. FORMULAE INVOLVING T-MATRICES, f- AND d-TENSORS

T^a are the matrix representatives of the generators in a representation R_x of a semi-simple Lie group G. They obey the commutation rule:

$$[T^a, T^b] = i f^{abc} T^c \qquad \ldots (B1)$$

where the f^{abc} are the structure constants of the group, and are normalized according to:

$$Tr\,(T^a T^b) = T_x \delta^{ab} \qquad \ldots (B2)$$

The value of the quadratic Casimir operator in the representation R_x is denoted by C_x, i.e.

$$C_x = T^a T^a \qquad \ldots (B3)$$

For example, if the gauge group is $SU(n)$ then

a) in the adjoint representation:

$$C_A = n \qquad \ldots (B4a)$$

$$T_A = n \qquad \ldots (B4b)$$

b) in the fundamental (quark) representation:

$$C_F = \frac{n^2 - 1}{2n} \qquad \ldots (B5a)$$

$$T_F = \frac{1}{2} \qquad \ldots (B5b)$$

The former representation is by necessity adopted for the gauge vector meson and ghost fields. The latter is frequently adopted for the scalar meson and fermion fields.

In what follows, refs. [1] and [2] have been the sources of basic properties.

The matrices T obey the Jacobi identity:

$$[[T^a, T^b], T^c] + [[T^b, T^c], T^a] + [[T^c, T^a], T^b] = 0 \qquad \ldots (B6)$$

which can be expressed as a relation amongst the structure constants:

$$f^{abd} f^{dce} + f^{bcd} f^{dae} + f^{cad} f^{dbe} = 0 \qquad \ldots (B7)$$

Also

$$f^{abc} f^{abd} = C_A \delta^{cd}$$
$$= n \; \delta^{cd} \; \text{in} \; SU(n) \qquad \ldots(B8)$$

In the fundamental representation, for which the notation $T^a = \frac{1}{2}\lambda^a$ is used by some authors, the matrices obey a multiplication law:

$$T^a T^b = \frac{1}{2}\left(d^{abc} + i f^{abc}\right) T^c + \frac{1}{2n} \delta^{ab} \qquad \ldots(B9)$$

leading to the anti-commutation rule:

$$\{ T^a, T^b \} = d^{abc} T^c + \frac{1}{n} \delta^{ab} \qquad \ldots(B10)$$

The totally symmetric and anti-symmetric natures of the tensors d^{abc} and f^{abc} respectively are expressed in the derived relations:

$$\frac{1}{2} d^{abc} = Tr \, \{ T^a, T^b \} \, T^c \qquad \ldots(B11)$$

$$\frac{1}{2} i f^{abc} = Tr \, [T^a, T^b] \, T^c \qquad \ldots(B12)$$

Further useful formulae are:

$$d^{abc} f^{abd} = 0 \qquad \ldots(B13)$$

$$d^{abc} d^{abd} = \frac{n^2 - 4}{n} \delta^{cd} \qquad \ldots(B14)$$

and also

$$f^{abc} d^{cde} + f^{dbc} d^{ace} + f^{ebc} d^{abc} = 0 \qquad \ldots(B15)$$

It is now possible to derive the following:

$$\begin{aligned} T^a T^b T^a &= \left[\frac{1}{2}\left(d^{abc} + i f^{abc}\right) T^c + \frac{1}{2n} \delta^{ab}\right] T^a \\ &= \frac{1}{4}\left(d^{abc} + i f^{abc}\right)\left(d^{cad} + i f^{cad}\right) T^d + \frac{1}{2n} T^b \\ &= \frac{1}{4}\left[\left(\frac{n^2-4}{n}\right)\delta^{bd} - n \, \delta^{bd}\right] T^d + \frac{1}{2n} T^b \\ &= -\frac{1}{2n} T^b \end{aligned} \qquad \ldots(B16)$$

$$\begin{aligned} i f^{abc} T^b T^c &= -i f^{acb} T^b T^c \\ &= -[T^a, T^c] \, T^c \\ &= - T^a T^c T^c + T^c T^a T^c \\ &= -\frac{n}{2} T^a \end{aligned} \qquad \ldots(B17)$$

$$
\begin{aligned}
d^{abc}T^bT^c &= d^{abc}\left[\tfrac{1}{2}(d^{bcd}+if^{bcd})T^d+\tfrac{1}{2n}\delta^{bc}\right] \\
&= \tfrac{1}{2}d^{abc}d^{bcd}T^d \\
&= \frac{n^2-4}{2n}\,T^a
\end{aligned}
\qquad \text{...(B18)}
$$

It must be remembered that (B9) ⟶ (B18) apply only for the fundamental representation in $SU(n)$.

One further result for the structure constants is that for the three-fold multiple $(f^3)^{abc} = f^{aef}f^{cde}f^{bfd}$

Equation (B7) can be re-expressed as:

$$
f^{abe}f^{cde} + f^{ace}f^{bde} + f^{ade}f^{bce} = 0 \qquad \text{...(B19)}
$$

Multiplying (B19) on both sides by f^{bdf} gives

$$
(f^3)^{abc} - f^{ace}C_A\delta^{ef} + (f^3)^{acf} = 0
$$

So that

$$
\begin{aligned}
(f^3)^{abc} &= \tfrac{1}{2}C_A f^{abc} \\
&= \tfrac{n}{2}f^{abc} \quad \text{in } SU(n)
\end{aligned}
\qquad \text{...(B20)}
$$

APPENDIX C. THE GENERAL GAUGE PROPAGATOR

Here we derive the gauge field propagator in a gauge specified by parameters a^2 and b^2, where the special cases are:

1) $a^2 = b^2 \equiv \alpha^{-1/4}$: covariant gauges.

2) $a^2 = 0$: a class of axial gauges.
$b^2 \to \infty$ yields the usual axial gauge.

3) $b^2 = 0$: a class of coulomb gauges.
$a^2 \to \infty$ yields the usual coulomb gauge.

The gauge fixing term is

$$-\tfrac{1}{2}\left[C(x)\right]^2 = -\tfrac{1}{2}\left[D^\mu A_\mu(x)\right]^2 \qquad \text{...(C1)}$$

with

$$D_\mu = a^2\partial_\mu + (b^2-a^2)\frac{\partial^\nu n_\nu}{n^2} n_\mu \qquad \text{...(C2)}$$

For the A-field, the path integral technique yields the following integral for the interaction free part:

$$\int dA_\mu\, e^{\frac{i}{2}\left[A^\mu K_{\mu\nu}A^\nu + A^\nu J_\nu\right]} = \left(\begin{smallmatrix}\text{some normalising}\\ \text{factor}\end{smallmatrix}\right) e^{-\frac{i}{2} J^\mu K^{-1}_{\mu\nu} J^\nu} \qquad \text{...(C3)}$$

where $-K^{-1}_{\mu\nu}$ is the propagator.

In momentum space, the free Lagrangian and gauge fixing term provide the matrix

$$\begin{aligned} K_{\mu\nu} &= k^2\eta_{\mu\nu} - k_\mu k_\nu + \left[a^2 k + (b^2-a^2)\frac{k.n}{n^2}n_\mu\right]\left[a^2 k_\nu + (b^2-a^2)\frac{k.n}{n^2}n_\nu\right] \\ &= u\, k_\mu k_\nu + v\, n_\mu n_\nu + \omega(k_\mu n_\nu + n_\mu k_\nu) + x\,\eta_{\mu\nu} \end{aligned} \qquad \text{...(C4)}$$

with

$$\left.\begin{aligned} u &= a^4 - 1 \\ v &= (b^2-a^2)\frac{(k.n)^2}{n^4} \\ \omega &= a^2(b^2-a^2)\frac{k.n}{n^2} \\ x &= k^2 \end{aligned}\right\} \qquad \text{...(C5)}$$

Letting

$$K^{-1}_{\mu\nu} = u' k_\mu k_\nu + v' n_\mu n_\nu + \omega'(k_\mu n_\nu + n_\mu k_\nu) + x'\eta_{\mu\nu} \qquad \text{...(C6)}$$

we have

$$K_\mu{}^\sigma K^{-1}_{\sigma\nu} = \eta_{\mu\nu}$$

$$= [uu'k^2 + \omega\omega' n^2 + (u\omega' + \omega u')k.n + ux' + xu']k_\mu k_\nu$$
$$+ [\omega\omega' k^2 + vv'n^2 + (v\omega' + \omega v')k.n + vx' + xv']n_\mu n_\nu$$
$$+ [u\omega' k^2 + \omega v' n^2 + (uv' + \omega\omega')k.n + \omega x' + x\omega']k_\mu n_\nu$$
$$+ [\omega u' k^2 + v\omega' n^2 + (vu' + \omega\omega')k.n + \omega x' + x\omega']n_\mu k_\nu$$
$$+ xx'\eta_{\mu\nu} \qquad \text{...(C7)}$$

So, equating coefficients of $k_\mu k_\nu$, $n_\mu n_\nu$, etc., and using Cramer's rule on the resulting simultaneous equations, we arrive at:

$$\left.\begin{aligned} u' &= \mathcal{D}^{-1}\left[-\tfrac{u}{x}(n^2 v + k.n\,\omega + x) + \tfrac{\omega}{x}(k.n\,u + n^2\omega)\right] \\ v' &= \mathcal{D}^{-1}\left[-\tfrac{v}{x}(k^2 u + k.n\,\omega + x) + \tfrac{\omega}{x}(k.n\,v + k^2\omega)\right] \\ \omega' &= \mathcal{D}^{-1}\left[-\tfrac{\omega}{x}(k^2 u + k.n\,\omega + x) + \tfrac{u}{x}(k.n\,v + k^2\omega)\right] \\ x' &= \tfrac{1}{x} \end{aligned}\right\} \qquad \text{...(C8)}$$

where

$$\mathcal{D} = (k^2 u + k.n\,\omega + x)(n^2 v + k.n\,\omega + x) - (k.n\,u + n^2\omega)(k.n\,v + k^2\omega) \qquad \text{...(C9)}$$

Substituting for u, v, ω and x in (C8) and (C9) gives

$$\left.\begin{aligned} u' &= \mathcal{D}^{-1}\left[1 - a^4 + (b^2-a^2)^2\frac{(k.n)^2}{k^2n^2}\right] \\ v' &= 0 \\ \omega' &= \mathcal{D}^{-1}\left[-a^2(b^2-a^2)\frac{k.n}{n^2} - (b^2-a^2)^2\frac{(k.n)^3}{k^2n^4}\right] \\ x' &= \frac{1}{k^2} \end{aligned}\right\} \qquad \text{...(C10)}$$

where $\mathcal{D} = \left[a^2k^2 + (b^2-a^2)\dfrac{(k.n)^2}{k^2}\right]^2 \qquad \text{...(C11)}$

yielding the propagator

$$\Delta_{\mu\nu}(k) = -K^{-1}{}_{\mu\nu}(k)$$
$$= -\frac{1}{k^2}\left\{\eta_{\mu\nu} - \frac{(b^2-a^2)\frac{k.n}{n^2}}{\left[a^2k^2 + (b^2-a^2)\frac{(k.n)^2}{n^2}\right]}(k_\mu n_\nu + n_\mu k_\nu)\right.$$
$$\left. + \frac{(1-a^4)k^2 + (b^2-a^2)^2\frac{(k.n)^2}{n^2}}{\left[a^2k^2 + (b^2-a^2)\frac{(k.n)^2}{n^2}\right]^2}k_\mu k_\nu\right\} \qquad \text{...(C12)}$$

The special cases are:

Covariant gauges $\quad a^2 = b^2 \equiv c^{-1}$

$$\Delta_{\mu\nu}(k) = -\frac{1}{k^2}\left[\eta_{\mu\nu} - \frac{(1-c)}{k^2} k_\mu k_\nu\right] \qquad \ldots(C13)$$

Axial gauges $\quad a^2 = 0$

$$\Delta_{\mu\nu}(k) = -\frac{1}{k^2}\left[\eta_{\mu\nu} - \frac{1}{k.n}(k_\mu n_\nu + n_\mu k_\nu) + \frac{n^4 k^2 + b^4 n^2 (k.n)^2}{b^4 (k.n)^4} k_\mu k_\nu\right] \qquad \ldots(C14)$$

Coulomb gauges $\quad b^2 = 0$

$$\Delta_{\mu\nu}(k) = -\frac{1}{k^2}\left[\eta_{\mu\nu} + \frac{k.n}{n^2 k^2 - (k.n)^2}(k_\mu n_\nu + n_\mu k_\nu) + \frac{(1-a^4) n^4 k^2 + a^4 n^2 (k.n)^2}{a^4 [n^2 k^2 - (k.n)^2]^2} k_\mu k_\nu\right] \qquad \ldots(C15)$$

For reasons of calculational convenience we choose n_μ to be a unit timelike vector, $n_\mu = (1,0,0,0)$. In this case the general gauge propagator becomes:

$$\Delta_{\mu\nu}(k) = -\frac{1}{k^2}\left[\eta_{\mu\nu} + \frac{(b^2-a^2) k_0}{(b^2 k_0^2 - a^2 \underline{k}^2)}(k_\mu \eta_{\nu 0} + \eta_{\mu 0} k_\nu) + \frac{(1-a^4) k^2 + (b^2-a^2)^2 k_0^2}{(b^2 k_0^2 - a^2 \underline{k}^2)^2} k_\mu k_\nu\right] \qquad \ldots(C16)$$

APPENDIX D. LISTS OF INTEGRALS

In the course of one-loop calculations we encounter integrals of the form

$$I_N = \int \frac{d^4q \; q_0^{2r}\, \mathbf{q}^{2s}}{q_0^{2t}\, \mathbf{q}^{2u}\, q^{2m}} \qquad \text{...(D1)}$$

where $r + s = t + u + m - 2$

and

$$J_N = \int \frac{d^4q \; q_0^{2r}\, \mathbf{q}^{2s}}{q^{2m}[\,b^2 q_0^2 - a^2 \mathbf{q}^2\,]^n} \qquad \text{...(D2)}$$

where $r + s = m + n - 2$

We evaluate these by contour integration in the q_0 plane followed by integration over the three-momentum $\mathbf{q}$ using ultraviolet and infrared cut-offs Λ and μ respectively. A principal value prescription is required to deal with the poles at $q_0 = 0$ and $b^2 q_0^2 - a^2 \mathbf{q}^2 = 0$. In the former case this is achieved by setting

$$\frac{1}{q_0} = \frac{1}{2}\left[\frac{1}{q_0 + i\epsilon} + \frac{1}{q_0 - i\epsilon}\right] \qquad \text{...(D3)}$$

and in the latter by averaging over $f(a)$ and $f(-a)$ in the final result.

To better illustrate this process, two simple examples are chosen:

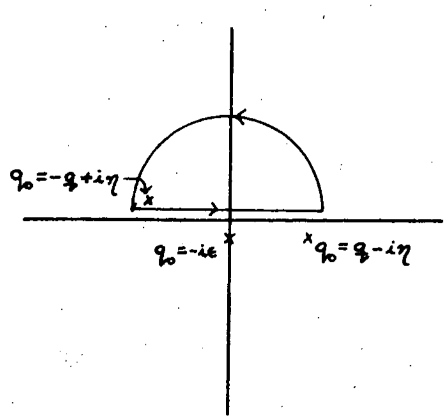
1) $$I_7 = \int \frac{d^4q}{q_0^2 q^2} \qquad \text{...(D4)}$$

Using the P.V.P. this is

$$I_7 = \int d^3\mathbf{q} \int dq_0 \, \frac{1}{2}\left[\frac{1}{(q_0+i\epsilon)^2} + \frac{1}{(q_0-i\epsilon)^2}\right] \frac{1}{(q_0 - |\mathbf{q}| + i\eta)(q_0 + |\mathbf{q}| - i\eta)} \qquad \text{...(D5)}$$

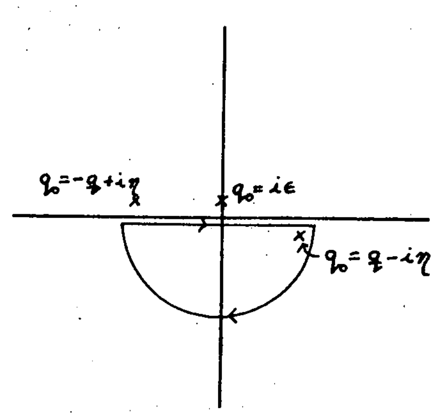
where ϵ and η are arbitrarily small, positive quantities.

The poles in the q_0 plane are therefore given as:



(a)

and



(b)

Upon closing the contour above in (a) and below in (b) a straightforward calculation gives

$$I_7 = 2\pi i \int d^3q \left(\frac{-1}{2|q|^3}\right) \qquad \ldots (D6)$$

After transforming to spherical polar coordinates and performing the angular integrations we are left with

$$I_7 = -4i\pi^2 \int_0^\infty \frac{d|q|}{|q|}$$
$$= -2i\pi^2 \ln \frac{\Lambda^2}{\mu^2} \qquad \ldots (D7)$$

where Λ and μ are cut-offs.

I integrals are listed in Table D1.

2) $$J_3 = \int \frac{d^4q \; q^2}{q^2 [b^2 q_0^2 - a^2 q^2]^2} \qquad \ldots (D8)$$

Firstly, we use the Feynman Parameter formula [3] :

$$\frac{1}{A^m B^n} = \frac{(m+n-1)!}{(m-1)!\,(n-1)!} \int_0^1 dx\, (1-x)^{m-1}\, x^{n-1} \frac{1}{[A(1-x)+Bx]^{m+n-1}} \qquad \ldots (D9)$$

to combine denominators. In this case $m=1, n=2$.

$$J_3 = \frac{2}{a^4} \int d^4q \int_0^1 dx \;\; x \;\; \frac{q^2}{[q_0^2(1-\frac{a^2-b^2}{a^2}x) - q^2]^3} \qquad \ldots (D10)$$

Now the integration variable q_0 is shifted by $[1-\frac{a^2-b^2}{a^2}x]^{1/2}$ to give

$$J_3 = \frac{2}{a^4} \int_0^1 dx\, x\, [1-\frac{a^2-b^2}{a^2}x]^{3/2} \int \frac{d^4q\; q^2}{q^6} \qquad \ldots (D11)$$

All that remains is to evaluate the integral over x by parts, and then substitute for the q-integral the result from Table D1,

(i.e. I_2):

$$J_3 = \frac{2}{a^4} \cdot \frac{2a^4}{3(a^2-b^2)^2} \cdot \left[2 + (b^2-3a^2)\frac{b}{a^3}\right]\left(-\frac{3}{4}\right) i\pi^2 \ln \frac{\Lambda^2}{\mu^2}$$
$$= \frac{-(2a^3-3a^2b+b^3)}{a^3(a^2-b^2)^2} \cdot i\pi^2 \ln \frac{\Lambda^2}{\mu^2}$$
$$= \frac{-(2a+b)}{a^3(a+b)^2} \cdot i\pi^2 \ln \frac{\Lambda^2}{\mu^2} \qquad \ldots (D12)$$

If $a=b$, the result is straightforward. When $a \neq b$ a P.V.P. is required to deal with the $q_0^2 = \frac{b^2}{a^2} q^2$ singularity.

A prescription which corresponds to the usual one in the special cases is to average over $J_3(a)$ and $J_3(-a)$.

J integrals are listed in Table D2.

It is worth remarking that once I_1 and I_7 are evaluated, it is easy to find I_8. Similarly, once J_3 is known, so are $J_{15} = \frac{1}{2} \frac{\partial}{\partial a^2} J_3$, $J_{16} = -\frac{1}{2} \frac{\partial}{\partial b^2} J_3$ and so on.

Thus, it is not necessary to individually work out each I and J integral.

I - INTEGRALS

	Integrand	Factor		Integrand	Factor
I_1	$\frac{1}{q^4}$	1	I_{12}	$\frac{q_0^2}{\not{q}^2 q^4}$	-1
I_2	$\frac{\not{q}^2}{q^6}$	$-\frac{3}{4}$	I_{13}	$\frac{q_0^4}{\not{q}^2 q^6}$	$-\frac{3}{4}$
I_3	$\frac{q_0^2}{q^6}$	$\frac{1}{4}$	I_{14}	$\frac{q_0^6}{\not{q}^2 q^8}$	$-\frac{5}{8}$
I_4	$\frac{\not{q}^4}{q^8}$	$\frac{5}{8}$	I_{15}	$\frac{q_0^4}{\not{q}^4 q^4}$	-3
I_5	$\frac{q_0^2 \not{q}^2}{q^8}$	$-\frac{1}{8}$	I_{16}	$\frac{\not{q}^6}{q^{10}}$	$-\frac{35}{64}$
I_6	$\frac{q_0^4}{q^8}$	$\frac{1}{8}$	I_{17}	$\frac{q_0^2 \not{q}^4}{q^{10}}$	$\frac{5}{64}$
I_7	$\frac{1}{q_0^2 q^2}$	-2	I_{18}	$\frac{q_0^4 \not{q}^2}{q^{10}}$	$-\frac{3}{64}$
I_8	$\frac{\not{q}^2}{q_0^2 q^4}$	3	I_{19}	$\frac{q_0^6}{q^{10}}$	$\frac{5}{64}$
I_9	$\frac{\not{q}^4}{q_0^2 q^6}$	$-\frac{15}{4}$	I_{20}	$\frac{\not{q}^8}{q^{12}}$	$\frac{63}{128}$
I_{10}	$\frac{\not{q}^6}{q_0^2 q^8}$	$\frac{35}{8}$	I_{21}	$\frac{q_0^4 \not{q}^4}{q^{12}}$	$\frac{3}{128}$
I_{11}	$\frac{1}{\not{q}^2 q^2}$	-2	I_{22}	$\frac{q_0^8}{q^{12}}$	$\frac{7}{128}$

Table D1. Integrals of the form $I_N = \int \frac{d^4q \; q_0^{2r} \not{q}^{2s}}{q_0^{2t} \not{q}^{2u} q^{2m}}$ expressed as a multiple ('factor') of the divergent constant $i \pi^2 \ln \frac{\Lambda^2}{\mu^2}$.

J - INTEGRALS

	Integrand	Factor		Integrand	Factor
J_1	$\frac{1}{[b^2q_0^2-a^2\underline{q}^2]^2}$	$\frac{1}{a^3 b}$	J_{13}	$\frac{\underline{q}^2}{[b^2q_0^2-a^2\underline{q}^2]^3}$	$-\frac{3}{4a^5 b}$
J_2	$\frac{1}{q^2[b^2q_0^2-a^2\underline{q}^2]}$	$\frac{2}{a(a+b)}$	J_{14}	$\frac{q_0^2}{[b^2q_0^2-a^2\underline{q}^2]^3}$	$\frac{1}{4a^3 b^3}$
J_3	$\frac{\underline{q}^2}{q^2[b^2q_0^2-a^2\underline{q}^2]^2}$	$-\frac{2a+b}{a^3(a+b)^2}$	J_{15}	$\frac{\underline{q}^4}{q^2[b^2q_0^2-a^2\underline{q}^2]^3}$	$\frac{8a^2+9ab+3b^2}{4a^5(a+b)^3}$
J_4	$\frac{q_0^2}{q^2[b^2q_0^2-a^2\underline{q}^2]^2}$	$\frac{1}{ab(a+b)^2}$	J_{16}	$\frac{q_0^2\,\underline{q}^2}{q^2[b^2q_0^2-a^2\underline{q}^2]^3}$	$-\frac{3a+b}{4a^3b(a+b)^3}$
J_5	$\frac{\underline{q}^2}{q^4[b^2q_0^2-a^2\underline{q}^2]}$	$-\frac{a+2b}{a(a+b)^2}$	J_{17}	$\frac{q_0^4}{q^2[b^2q_0^2-a^2\underline{q}^2]^3}$	$\frac{a+3b}{4ab^3(a+b)^3}$
J_6	$\frac{q_0^2}{q^4[b^2q_0^2-a^2\underline{q}^2]}$	$\frac{1}{(a+b)^2}$	J_{18}	$\frac{\underline{q}^4}{[b^2q_0^2-a^2\underline{q}^2]^4}$	$\frac{5}{8a^7 b}$
J_7	$\frac{\underline{q}^4}{q^6[b^2q_0^2-a^2\underline{q}^2]}$	$\frac{3a^2+9ab+8b^2}{4a(a+b)^3}$	J_{19}	$\frac{q_0^2\,\underline{q}^2}{[b^2q_0^2-a^2\underline{q}^2]^4}$	$-\frac{1}{8a^5 b^3}$
J_8	$\frac{q_0^2\,\underline{q}^2}{q^6[b^2q_0^2-a^2\underline{q}^2]}$	$-\frac{a+3b}{4(a+b)^3}$	J_{20}	$\frac{q_0^4}{[b^2q_0^2-a^2\underline{q}^2]^4}$	$\frac{1}{8a^3 b^5}$
J_9	$\frac{q_0^4}{q^6[b^2q_0^2-a^2\underline{q}^2]}$	$\frac{3a+b}{4(a+b)^3}$	J_{21}	$\frac{\underline{q}^6}{[b^2q_0^2-a^2\underline{q}^2]^5}$	$-\frac{35}{64a^9 b}$
J_{10}	$\frac{\underline{q}^4}{q^4[b^2q_0^2-a^2\underline{q}^2]^2}$	$\frac{a^2+3ab+b^2}{a^3(a+b)^3}$	J_{22}	$\frac{q_0^2\,\underline{q}^4}{[b^2q_0^2-a^2\underline{q}^2]^5}$	$\frac{5}{64a^7 b^3}$
J_{11}	$\frac{q_0^2\,\underline{q}^2}{q^4[b^2q_0^2-a^2\underline{q}^2]^2}$	$-\frac{1}{a(a+b)^3}$	J_{23}	$\frac{q_0^4\,\underline{q}^2}{[b^2q_0^2-a^2\underline{q}^2]^5}$	$-\frac{3}{64a^5 b^5}$
J_{12}	$\frac{q_0^4}{q^4[b^2q_0^2-a^2\underline{q}^2]^2}$	$\frac{1}{b(a+b)^3}$	J_{24}	$\frac{q_0^6}{[b^2q_0^2-a^2\underline{q}^2]^5}$	$\frac{5}{64a^3 b^7}$

Table D2. Integrals of the form $J_N = \int \frac{d^4q\; q_0^{2r}\,\underline{q}^{2s}}{q^{2m}[b^2q_0^2 - a^2\underline{q}^2]^n}$ expressed as a multiple ('factor') of the divergent constant $i\pi^2 \ln \frac{\Lambda^2}{\mu^2}$. a and b are arbitrary constants. Note that in those factors involving $(a+b)$ in the denominator, $(a-b)$ has been divided through.

APPENDIX E. FORMULAE INVOLVING DIRAC MATRICES

The Dirac matrices obey the anticommutation algebra:

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} \qquad \text{...(E1)}$$

From (E1) we can derive the following properties of the matrices alone:

$$\gamma^\mu \gamma_\mu = 4 \qquad \text{...(E2)}$$

$$\gamma^\mu \gamma^\nu \gamma_\mu = -2\gamma^\nu \qquad \text{...(E3)}$$

$$\gamma_i \gamma_j = -2\delta_{ij} - \gamma_j \gamma_i \qquad \text{...(E4)}$$

$$\gamma_i \gamma_i = \underline{\gamma}^2 = -3 \qquad \text{...(E5)}$$

$$\gamma_i \gamma_0 \gamma_j = -\gamma_0 \gamma_i \gamma_j \qquad \text{...(E6)}$$

$$\gamma_i \gamma_0 \gamma_i = 3\gamma_0 \qquad \text{...(E7)}$$

$$\gamma_i \gamma_\ell \gamma_i = \gamma_\ell \qquad \text{...(E8)}$$

$$\gamma_0 \gamma_0 = \gamma_0^2 = 1 \qquad \text{...(E9)}$$

$$\gamma_0 \gamma_\ell \gamma_0 = -\gamma_\ell \qquad \text{...(E10)}$$

Also

$$\begin{aligned}\gamma_\mu \gamma_\rho \gamma_\lambda \gamma_\sigma \gamma_\nu = {} & 4\eta_{\lambda\sigma}\eta_{\rho\nu}\gamma_\mu - 2\eta_{\lambda\sigma}\gamma_\mu\gamma_\nu\gamma_\rho \\ & -2\eta_{\lambda\nu}\gamma_\mu\gamma_\rho\gamma_\sigma + 2\eta_{\sigma\nu}\gamma_\mu\gamma_\rho\gamma_\lambda \\ & -2\eta_{\rho\nu}\gamma_\mu\gamma_\sigma\gamma_\lambda + \gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_\lambda\end{aligned} \qquad \text{...(E11)}$$

There are also the following trace theorems [4] :

$$\mathrm{Tr}\ 1 = 4 \qquad \text{...(E12)}$$

$$\mathrm{Tr}\ \gamma^\mu = 0 \qquad \text{...(E13)}$$

$$\mathrm{Tr}\,(\gamma^{\mu_1}\gamma^{\mu_2}\ldots\gamma^{\mu_n}) = 0 \qquad \text{...(E14)}$$

$$\mathrm{Tr}\,(\gamma^\mu\gamma^\nu) = 4\eta^{\mu\nu} \qquad \text{...(E15)}$$

$$\mathrm{Tr}\,(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\sigma\nu} + \eta^{\nu\rho}\eta^{\sigma\mu}) \qquad \text{...(E16)}$$

From (E1)$\rightarrow$(E11) we can derive the following formulae involving the γ-matrices, vectors k_μ, p_μ and scalar m:

$$\not{k}\,\not{k} = k^2 \qquad \ldots(E17)$$

$$(\underline{\gamma}.\underline{k})^2 = -\underline{k}^2 \qquad \ldots(E18)$$

$$\gamma_\mu \not{k}\, \gamma_\nu = 2\,k_\mu \gamma_\nu - \not{k}\, \gamma_\mu \gamma_\nu \qquad \ldots(E19)$$

$$\gamma_0 \not{k}\, \gamma_0 = \gamma_0 k_0 + \underline{\gamma}.\underline{k} \qquad \ldots(E20)$$

$$\gamma_0 \not{k}\, \gamma_n + \gamma_n \not{k}\, \gamma_0 = 2\,k_0 \gamma_n + 2\,\gamma_0 k_n \qquad \ldots(E21)$$

$$\gamma_m \not{k}\, \gamma_n = -\not{k}\, \gamma_m \gamma_n + 2\,k_m \gamma_n \qquad \ldots(E22)$$

$$\gamma_i \not{k}\, \gamma_i = 3\,\gamma_0 k_0 - \underline{\gamma}.\underline{k} \qquad \ldots(E23)$$

$$\underline{\gamma}.\underline{k}\, \not{k}\, \underline{\gamma}.\underline{k} = \underline{k}^2 (\gamma_0 k_0 + \underline{\gamma}.\underline{k}) \qquad \ldots(E24)$$

$$\underline{\gamma}.\underline{k}\, \not{p}\, \underline{\gamma}.\underline{k} = \underline{k}^2 \not{p} + 2\,\underline{k}.\underline{p}\; \underline{\gamma}.\underline{k} \qquad \ldots(E25)$$

$$\gamma^\mu (\not{p} + \not{k} + m)\gamma_\mu = -2(\not{p} + \not{k}) + 4m \qquad \ldots(E26)$$

$$\not{k}\,(\not{p} + \not{k} + m)\not{k} = (k^2 + 2k.p)\not{k} - (\not{p} - m)k^2 \qquad \ldots(E27)$$

$$\gamma^\mu \not{k}\, \gamma_\lambda \not{k}\, \gamma_\mu = -4\,k_\lambda \not{k} + 2\,k^2 \gamma_\lambda \qquad \ldots(E28)$$

$$\gamma_0 \not{k}\, \gamma_0 \not{k}\, \gamma_0 = (k_0^2 + \underline{k}^2)\,\gamma_0 + 2\,k_0 \underline{\gamma}.\underline{k} \qquad \ldots(E29)$$

$$\gamma_0 \not{k}\, \gamma_\ell \not{k}\, \gamma_0 = k^2 \gamma_\ell + 2(\gamma_0 k_0 + \underline{\gamma}.\underline{k})k_\ell \qquad \ldots(E30)$$

$$\gamma_i \not{k}\, \gamma_0 \not{k}\, \gamma_j = [-\gamma_0(k_0^2 + \underline{k}^2) + 2\,k_0 \underline{\gamma}.\underline{k}\,]\,\gamma_i \gamma_j + 4\,k_0 k_i k_j \qquad \ldots(E31)$$

$$\gamma_i \not{k}\, \gamma_0 \not{k}\, \gamma_i = 3\,\gamma_0(k_0^2 + \underline{k}^2) - 2\,\gamma_0 \underline{\gamma}.\underline{k} \qquad \ldots(E32)$$

$$\underline{\gamma}.\underline{k}\, \not{k}\, \gamma_0 \not{k}\, \underline{\gamma}.\underline{k} = \gamma_0 \underline{k}^2 (k_0^2 + \underline{k}^2) + 2\,k_0 \underline{k}^2 \underline{\gamma}.\underline{k} \qquad \ldots(E33)$$

$$\not{k}\, \gamma_\ell \not{k} = (-k^2 + 2\not{k})\,\gamma_\ell \qquad \ldots(E34)$$

$$\gamma_i \not{k}\, \gamma_\ell \not{k}\, \gamma_i = -k^2 \gamma_\ell + (6\,\gamma_0 k_0 - 2\,\underline{\gamma}.\underline{k})\,k_\ell \qquad \ldots(E35)$$

$$\underline{\gamma}.\underline{k}\, \not{k}\, \gamma_\ell \not{k}\, \underline{\gamma}.\underline{k} = 2(k_0^2 \underline{\gamma}.\underline{k} + \underline{k}^2 \gamma_0 k_0)\,k_\ell - k^2 \underline{k}^2 \gamma_\ell \qquad \ldots(E36)$$

$$(\not{p} - m)\not{k}\, \gamma_0 + \gamma_0 \not{k}\,(\not{p} - m) = 2[p_0(\gamma_0 k_0 + \underline{\gamma}.\underline{k}) - k_0(\underline{\gamma}.\underline{p} + m) - \gamma_0 \underline{p}^2] \qquad \ldots(E37)$$

APPENDIX F. OFTEN USED IDENTITIES AND TRICKS

The following identities are used extensively in Appendices G and H:

$$\int d^4q\, f(q^2)\, q_\mu q_\nu = \frac{1}{4}\eta_{\mu\nu}\int d^4q\, f(q^2)\, q^2 \qquad \ldots\text{(F1)}$$

leading to

$$\int d^4q\, f(q^2)(q\cdot p)^2 = \frac{1}{4}p^2\int d^4q\, f(q^2)\, q^2$$

$$\int d^4q\, f(q^2)(q\cdot p)\, q_\lambda = \frac{1}{4}p_\lambda\int d^4q\, f(q^2)\, q^2$$

$$\int d^4q\, f(q^2)(q\cdot p)\, \not{q} = \frac{1}{4}\not{p}\int d^4q\, f(q^2)\, q^2$$

$$\int d^3q\, f(\underline{q}^2)\, q_m q_n = \frac{1}{3}\delta_{mn}\int d^3q\, f(\underline{q}^2)\, \underline{q}^2 \qquad \ldots\text{(F2)}$$

leading to

$$\int d^3q\, f(\underline{q}^2)(\underline{q}\cdot\underline{p})^2 = \frac{1}{3}\underline{p}^2\int d^3q\, f(\underline{q}^2)\, \underline{q}^2$$

$$\int d^3q\, f(\underline{q}^2)(\underline{q}\cdot\underline{p})\, q_\ell = \frac{1}{3}p_\ell\int d^3q\, f(\underline{q}^2)\, \underline{q}^2$$

$$\int d^3q\, f(\underline{q}^2)(\underline{q}\cdot\underline{p})(\underline{\gamma}\cdot\underline{q}) = \frac{1}{3}(\underline{\gamma}\cdot\underline{p})\int d^3q\, f(\underline{q}^2)\, \underline{q}^2$$

$$\int d^4q\, f(q^2)\, q_\mu q_\nu q_\lambda q_\kappa = \frac{1}{24}\left(\eta_{\mu\nu}\eta_{\lambda\kappa} + \eta_{\mu\lambda}\eta_{\nu\kappa} + \eta_{\mu\kappa}\eta_{\nu\lambda}\right)\int d^4q\, f(q^2)(q^2)^2 \qquad \ldots\text{(F3)}$$

whence

$$\int d^4q\, f(q^2)(q\cdot p)^4 = \frac{1}{8}(p^2)^2\int d^4q\, f(q^2)(q^2)^2$$

$$\int d^3q\, f(\underline{q}^2)\, q_m q_n q_\ell q_k = \frac{1}{15}\left(\delta_{mn}\delta_{\ell k} + \delta_{m\ell}\delta_{nk} + \delta_{mk}\delta_{n\ell}\right)\int d^3q\, f(\underline{q}^2)(\underline{q}^2)^2 \qquad \ldots\text{(F4)}$$

whence

$$\int d^3q\, f(\underline{q}^2)(\underline{q}\cdot\underline{p})^4 = \frac{1}{5}(\underline{p}^2)^2\int d^3q\, f(\underline{q}^2)(\underline{q}^2)^2$$

Elementary Ward identities provide the next two results:

$$k\cdot(2p+k) = [(p+k)^2-m^2]-(p^2-m^2) \qquad \text{...(F5)}$$

and

$$\not{k} = (\not{p}+\not{k}-m)-(\not{p}-m) \qquad \text{...(F6)}$$

There are also:

$$\begin{aligned} \not{k}\cdot\frac{1}{\not{p}+\not{k}-m} &= 1-(\not{p}-m)\cdot\frac{\not{k}}{k^2}+\cdots \\ \frac{1}{\not{p}+\not{k}-m}\cdot\not{k} &= 1-\frac{\not{k}}{k^2}\cdot(\not{p}-m)+\cdots \end{aligned} \qquad \text{...(F7)}$$

$$\begin{aligned} \frac{1}{\not{k}}\cdot\frac{1}{\not{p}+\not{k}-m}\cdot\frac{1}{\not{k}} &= \not{k}\cdot\left[\frac{1}{\not{k}}-\frac{1}{\not{k}}\cdot(\not{p}-m)\cdot\frac{1}{\not{k}}+\cdots\right]\not{k} \\ &= \not{k}-(\not{p}-m)+\cdots \end{aligned} \qquad \text{...(F8)}$$

and finally:

$$p^{\kappa}\left[(q-r)_{\kappa}\eta_{\lambda\mu}+(r-p)_{\lambda}\eta_{\mu\kappa}+(p-q)_{\mu}\eta_{\kappa\lambda}\right] = (r^2-q^2)\eta_{\lambda\mu}+q_{\lambda}q_{\mu}-r_{\lambda}r_{\mu} \qquad \text{...(F9)}$$

when $p+q+r = 0$.

APPENDIX G. DETAILS OF CALCULATIONS IN SPECIFIC GAUGES

First we list the diagrams contributing to the one-loop correction in each case, followed by the corresponding combinatorical coeffecients (written as C.C. [...]). Tadpole diagrams are not included. Seagull diagrams give contributions which are dimensionally regularised to zero. We only show this for the covariant gauge scalar meson self-energy.

SCALAR MESON SELF-ENERGY:

$$\Pi(p^2) = \Pi_{(i)}(p^2) + \Pi_{(ii)}(p^2) \qquad \ldots(G1)$$

$\Pi_{(ii)}(p^2)$ is the seagull diagram.

$$C.C.[\Pi_{(i)}(p)] = T^a T^b \delta^{ab}$$

$$= C_\phi \qquad \ldots(G2)$$

FERMION SELF-ENERGY:

$\Sigma_{(i)}(p)$ is the only diagram contributing.

$$C.C.[\Sigma_{(i)}(p)] = T^a T^b \delta^{ab}$$

$$= C_\psi \qquad \ldots(G3)$$

VECTOR MESON SELF-ENERGY:

$$\Pi^{aa'}_{\kappa\kappa'}(p) = \tfrac{1}{2}\left[\Pi^{aa'}_{\kappa\kappa'(i)}(p) + \Pi^{aa'}_{\kappa\kappa'(ii)}(p)\right] - \Pi^{aa'}_{\kappa\kappa'(iii)}(p) \qquad \ldots(G4)$$

$\Pi^{aa'}_{\kappa\kappa'(ii)}(p)$ is the seagull diagram.

$$C.C.[\Pi^{aa'}_{\kappa\kappa'(i)}(p)] = (-if^{abc})\,\delta^{bb'}\delta^{cc'}(-if^{ab'c'})$$

$$= -C_A\,\delta^{aa'} \qquad \ldots(G5)$$

$$C.C.[\Pi^{aa'}_{\kappa\kappa'(iii)}(p)] = (if^{abc})\,\delta^{bb'}\delta^{cc'}(if^{ab'c'})$$

$$= -C_A\,\delta^{aa'} \qquad \ldots(G6)$$

SCALAR FIELD CONTRIBUTION TO THE VECTOR MESON SELF-ENERGY:

$$\tilde{\Pi}^{aa'}_{\kappa\kappa'}(p) = \tilde{\Pi}^{aa'}_{\kappa\kappa'(i)}(p) + \tilde{\Pi}^{aa'}_{\kappa\kappa'(ii)}(p) \quad \text{...(G7)}$$

$\tilde{\Pi}^{aa'}_{\kappa\kappa'(ii)}(p)$ is the seagull diagram.

$$c.c.\,[\tilde{\Pi}^{aa'}_{\kappa\kappa'(i)}(p)] = Tr\,(T^a T^{a'})$$

$$= T_\phi\, \delta^{aa'} \quad \text{...(G8)}$$

FERMION FIELD CONTRIBUTION TO THE VECTOR MESON SELF-ENERGY:

$\Sigma^{aa'}_{(i)\kappa\kappa'}(p)$ is the only diagram contributing.

$$c.c.\,[\Sigma^{aa'}_{(i)\kappa\kappa'}(p)] = Tr\,(T^a T^{a'})$$

$$= T_\psi\, \delta^{aa'} \quad \text{...(G9)}$$

SCALAR VERTEX CORRECTION:

$$\Gamma^a_\lambda(p) = \Gamma^a_{\lambda(i)}(p) + \Gamma^a_{\lambda(ii)}(p) + \Gamma^a_{\lambda(iii)}(p) + \Gamma^a_{\lambda(iv)}(p) + \Gamma^a_{\lambda(v)}(p) \quad \text{...(G10)}$$

$$c.c.\,[\Gamma^a_{\lambda(i)}(p)] = T^b T^a T^c \delta^{bc}$$

$$= T^b T^a T^b \quad \text{...(G11)}$$

$$c.c.\,[\Gamma^a_{\lambda(ii)}(p)] = T^b (i f^{ab'c'} T^c)\, \delta^{bb'} \delta^{cc'}$$

$$= i f^{abc} T^b T^c \quad \text{...(G12)}$$

$$c.c.\,[\Gamma^a_{\lambda(iii)}(p)] = -i f^{ab'c'} \{T^{b'}, T^{c'}\}\, \delta^{bb'} \delta^{cc'}$$

$$= -i f^{abc} \left(d^{abe} T^e + \frac{1}{n} \delta^{bc}\right)$$

$$= 0 \quad \text{...(G13)}$$

So this diagram does not contribute.

$$c.c.\,[\Gamma^a_{\lambda(iv)+(v)}(p)] = T^b\{T^b, T^a\} + \{T^a, T^b\} T^b$$

$$= 2\,(T^b T^b T^a + T^b T^a T^b)$$

$$= 2\, T^b \{T^b, T^a\}$$

$$= 2\,\left(d^{abc} T^b T^c + \frac{1}{n} T^a\right) \quad \text{...(G14)}$$

FERMION VERTEX CORRECTION:

$$\Gamma^{a}_{\lambda}(p) = \Gamma^{a}_{\lambda(i)}(p) + \Gamma^{a}_{\lambda(ii)}(p) \qquad \text{...(G15)}$$

$$\begin{aligned} C.C.[\Gamma^{a}_{\lambda(i)}(p)] &= T^{b} T^{a} T^{c} \delta^{bc} \\ &= T^{b} T^{a} T^{b} \qquad \text{...(G16)} \end{aligned}$$

$$\begin{aligned} C.C.[\Gamma^{a}_{\lambda(ii)}(p)] &= T^{b} (i f^{ab'c'} T^{c}) \delta^{bb'} \delta^{cc'} \\ &= i f^{abc} T^{b} T^{c} \qquad \text{...(G17)} \end{aligned}$$

GHOST SELF-ENERGY:

$\Pi^{aa'}_{(i)}(p)$ is the only diagram contributing.

$$\begin{aligned} C.C.[\Pi^{aa'}_{(i)}(p)] &= i f^{abc} i f^{a'c'b'} \delta^{bb'} \delta^{cc'} \\ &= - f^{abc} f^{a'cb} \\ &= f^{abc} f^{a'bc} \\ &= C_{A} \delta^{aa'} \qquad \text{...(G18)} \end{aligned}$$

GHOST VERTEX CORRECTION:

$$\Gamma^{abc}_{\lambda}(p) = \Gamma^{abc}_{\lambda(i)}(p) + \Gamma^{abc}_{\lambda(ii)}(p) \qquad \text{...(G19)}$$

$$\begin{aligned} C.C.[\Gamma^{abc}_{\lambda(i)}(p)] &= i f^{dce} i f^{af'e} i f^{d'fb} \delta^{ee'} \delta^{aa'} \delta^{ff'} \\ &= -i f^{dce} f^{afe} f^{dfb} \\ &= i f^{afe} f^{bdf} f^{cde} \\ &= i (f^{3})^{abc} \\ &= \tfrac{1}{2} i C_{A} f^{abc} \qquad \text{...(G20)} \end{aligned}$$

Similarly

$$C.C.[\Gamma^{abc}_{\lambda(ii)}(p)] = \tfrac{1}{2} i C_{A} f^{abc} \qquad \text{...(G21)}$$

The resulting infinite corrections are expressed in terms of the divergent constant

$$L = \frac{g^2}{16\pi^2} \ln \frac{\Lambda^2}{\mu^2} \qquad \ldots(G22)$$

In this appendix and the next some operations are performed so often that no explicit comment is made. They are:

* The discarding of all terms which are convergent or odd, or which vanish under dimensional regularization.
* The use of equations (F1), (F2), (F3) and (F4).
* Taking the limit $\ell \to 2$ in the final step(s) of calculations.
* When it applies, the use of the symmetry of an integral under interchange of its integration variables.
* Making the replacement $r = -q - p$ in a calculation involving the function $\delta(p + q + r)$.

Some procedures are commented on when first used and then taken as understood in further applications of a similar nature.

COVARIANT GAUGES

SCALAR MESON SELF-ENERGY:

$$\Pi_{(ii)}(p^2)\Big|_\infty = 2ig^2 C_\phi \int \frac{d^{2l}k}{k^2}\, \eta^{\mu\nu}\left[-\eta_{\mu\nu} + (1-c)\frac{k_\mu k_\nu}{k^2}\right]\Big|_\infty$$
$$= -2ig^2 C_\phi \int \frac{d^{2l}k}{k^2}\,[2l-1+c]\Big|_\infty$$
$$= 0 \qquad \ldots(G23)$$

Thus

$$\Pi(p^2)\Big|_\infty = \Pi_{(i)}(p^2)\Big|_\infty$$
$$= -ig^2 C_\phi \int \frac{d^{2l}k}{k^2[(p+k)^2-m^2]}\,(2p+k)^\mu(2p+k)^\nu\left[-\eta_{\mu\nu}+(1-c)\frac{k_\mu k_\nu}{k^2}\right]\Big|_\infty \qquad \ldots(G24)$$

$$= -ig^2 C_\phi \int \frac{d^{2l}k}{k^2[(p+k)^2-m^2]}\left[-(2p+k)^2+(1-c)\frac{((2p+k).k)^2}{k^2}\right]\Big|_\infty$$
$$= -ig^2 C_\phi \int d^{2l}k\,\Big[-\frac{4p^2+4p.k+k^2}{k^2[(p+k)^2-m^2]} + \frac{(1-c)(p^2-m^2)^2}{k^4[(p+k)^2-m^2]}$$
$$- \frac{2(1-c)(p^2-m^2)}{k^4} + (1-c)\frac{(p^2-m^2)+2p.k+k^2}{k^4}\Big]\Big|_\infty$$
$$= -ig^2 C_\phi \int \frac{d^{2l}k}{k^4}\Big\{[-4p^2-4p.k-k^2]\left[1-\frac{2k.p}{k^2}-\frac{p^2-m^2}{k^2}+\frac{4(k.p)^2}{k^4}+\cdots\right]$$
$$-(1-c)(p^2-m^2)\Big\}\Big|_\infty$$

where (F5) has been used.

So,
$$\Pi(p^2)\Big|_\infty = -ig^2 C_\phi \int \frac{d^{2l}k}{k^4}\left[-2p^2-m^2-(1-c)(p^2-m^2)\right]$$
$$= -ig^2 C_\phi\left[-3p^2+c(p^2-m^2)\right](2\pi)^{-4} I_1$$
$$= \left[(c-3)p^2-m^2\right] C_\phi\, L \qquad \ldots(G25)$$

FERMION SELF-ENERGY:

$$\Sigma(p)\Big|_\infty = -ig^2 C_\psi \int \frac{d^{2l}k}{k^2[(p+k)^2-m]}\,\gamma^\mu(\not p+\not k+m)\gamma^\nu\left[-\eta_{\mu\nu}+(1-c)\frac{k_\mu k_\nu}{k^2}\right]\Big|_\infty \qquad \ldots(G26)$$

$$= -ig^2 C_\psi \int \frac{d^{2\ell}k}{k^2[(p+k)^2 - m^2]} \Big\{ 2(\not p + \not k) - 4m + \frac{(1-c)}{k^2}\big[(k^2 + 2p.k)\not k - k^2 \not p + m k^2\big]\Big\}\Big|_\infty$$

$$= -ig^2 C_\psi \int \frac{d^{2\ell}k}{k^4}\Big[1 - \frac{2k.p}{k^2} - \frac{p^2 - m^2}{k^2} + \frac{4(k.p)^2}{k^4} + \cdots\Big] \times$$

$$\times \Big[(1+c)\not p - (3+c)m + (3-c)\not k + 2(1-c)\frac{k.p}{k^2}\not k\Big]\Big|_\infty$$

where (E26) and (E27) have been used.

So $\Sigma(p)\big|_\infty = -ig^2 C_\psi \int \frac{d^{2\ell}k}{k^4}\Big[(1+c)\not p - (3+c)m - \frac{4k.p}{k^2}\not k\Big]$

$$= -ig^2 C_\psi [c\not p - (3+c)m](2\pi)^{-4} I_1$$

$$= [c\not p - (3+c)m]\, C_\psi L \qquad \ldots\text{(G27)}$$

VECTOR MESON SELF-ENERGY:

$$\Pi^{aa'}_{\kappa\kappa'}(p)\Big|_\infty = -ig^2 C_A \delta^{aa'} \int d^{2\ell}q\, d^{2\ell}r\; \delta(p+q+r) \times$$

$$\times \Big\{ \frac{-1}{2q^2r^2}\big[(q-r)_\kappa \eta_{\lambda\mu} + (r-p)_\lambda \eta_{\mu\kappa} + (p-q)_\mu \eta_{\kappa\lambda}\big]\big[(q-r)_{\kappa'}\eta_{\lambda'\mu'} + (r-p)_{\lambda'}\eta_{\mu'\kappa'} + (p-q)_{\mu'}\eta_{\kappa'\lambda'}\big] \times$$

$$\times \Big[-\eta^{\lambda\lambda'} + (1-c)\frac{q^\lambda q^{\lambda'}}{q^2}\Big]\Big[-\eta^{\mu\mu'} + (1-c)\frac{r^\mu r^{\mu'}}{r^2}\Big]$$

$$- \frac{q_\kappa r_{\kappa'}}{q^2 r^2} \Big\}\Big|_\infty \qquad \ldots\text{(G28)}$$

First we show that $\Pi^{aa'}_{\kappa\kappa'}(p)\Big|_\infty$ is doubly transverse:

$$p^\kappa p^{\kappa'} \Pi^{aa'}_{\kappa\kappa'}(p)\Big|_\infty = -ig^2 C_A \delta^{aa'} \int d^{2\ell}q\, d^{2\ell}r\; \delta(p+q+r) \times$$

$$\times \Big\{ \frac{1}{2q^2r^2}\big[(r^2 - q^2)\eta_{\lambda\mu} + q_\lambda q_\mu - r_\lambda r_\mu\big]\big[(r^2 - q^2)\eta_{\lambda'\mu'} + q_{\lambda'} q_{\mu'} - r_{\lambda'} r_{\mu'}\big] \times$$

$$\times \Big[-\eta^{\lambda\lambda'} + (1-c)\frac{q^\lambda q^{\lambda'}}{q^2}\Big]\Big[-\eta^{\mu\mu'} + (1-c)\frac{r^\mu r^{\mu'}}{r^2}\Big]$$

$$+ \frac{(p\cdot q)(p\cdot r)}{q^2 r^2} \Big\}\Big|_\infty \qquad \ldots\text{(G29)}$$

where (F9) has been used.

$$p^\kappa p^{\kappa'}\Pi^{aa'}_{\kappa\kappa'}(p)\Big|_\infty = -ig^2 C_A \delta^{aa'}\int \frac{d^{2\ell}q\, d^{2\ell}r}{2q^2 r^2}\ \delta(p+q+r)\Big\{\big[(r^2-q^2)\eta_{\lambda\mu}+q_\lambda q_\mu - r_\lambda r_\mu\big]\times$$
$$\times\Big(\big[(r^2-q^2)\eta^{\lambda\mu}+q^\lambda q^\mu - r^\lambda r^\mu\big] - 2\frac{(1-c)}{q^2}\big[r^2 q^\mu q^\lambda - (q\cdot r)q^\lambda r^\mu\big]$$
$$+(1-c)^2 (q\cdot r)(q^\lambda q^\mu - q^\lambda r^\mu)\Big[\frac{1}{q^2} - \frac{1}{r^2}\Big]\Big) + 2(p\cdot q)(p\cdot r)\Big\}\Big|_\infty$$
$$= -\tfrac{1}{2}ig^2 C_A \delta^{aa'}\int\frac{d^{2\ell}q\, d^{2\ell}r}{q^2 r^2}\ \delta(p+q+r)\big\{(2\ell-1)(q^4+r^4) + (4-4\ell)q^2r^2 +$$
$$-2(q\cdot r)^2 + 2(1-c)\frac{r^2}{q^2}\big[r^2q^2-(q\cdot r)^2\big] + 2(p\cdot q)(p\cdot r)\big\}\Big|_\infty$$
$$= -\tfrac{1}{2}ig^2 C_A\delta^{aa'}\int\frac{d^{2\ell}q}{q^4}\ \big\{-2(1-c)\big[p^2q^2-(q\cdot p)^2\big] +$$
$$+\Big[1-\frac{2q\cdot p}{q^2}-\frac{p^2}{q^2}+\frac{4(q\cdot p)^2}{q^4}+\cdots\cdots\Big]\times$$
$$\times\big[(2\ell-1)p^4 + (8\ell-6)p^2(q\cdot p) + (8\ell-8)(q\cdot p)^2 + 2q^2p^2\big]\big\}\Big|_\infty$$

where the replacement $r = -q - p$ has been made.

$$\text{So } p^\kappa p^{\kappa'}\Pi_{\kappa\kappa'}^{aa'}(p)\Big|_\infty = \tfrac{1}{2}ig^2 C_A\delta^{aa'}\int\frac{d^{2\ell}q}{q^4}\ \Big\{-p^4 + 20p^2\frac{(q\cdot p)^2}{q^2} - 32\frac{(q\cdot p)^4}{q^4}\Big\}$$
$$= 0 \qquad \ldots\text{(G30)}$$

Thus $\Pi^{aa'}_{\kappa\kappa'}(p)\big|_\infty$ is doubly transverse, of the form:

$$\Pi_{\kappa\kappa'}(p)\Big|_\infty = (-\eta_{\kappa\kappa'}p^2 + p_\kappa p_{\kappa'})\,\Pi_c \qquad \ldots\text{(G31)}$$

where Π_c is a scalar, and group indices have been suppressed. To determine Π_c we need only find

$$\Pi_\kappa^{\ \kappa}(p)\Big|_\infty = -3p^2\,\Pi_c \qquad \ldots\text{(G32)}$$

One could just as easily determine $\Pi_{00}(p)\big|_\infty$ or $\Pi_{kk}(p)\big|_\infty$. There is nothing special about the above choice. The appearance of the factor $(-3p^2)$ serves as a check on the calculation.

$$\Pi_\kappa{}^\kappa(p)\Big|_\infty = -\tfrac{1}{2} i g^2 C_A \int \frac{d^{2\ell}q\, d^{2\ell}r}{q^2 r^2}\, \delta(p+q+r) \times$$
$$\times \Big\{ \big[(q-r)_\kappa \eta_{\lambda\mu} + (r-p)_\lambda \eta_{\mu\kappa} + (p-q)_\mu \eta_{\kappa\lambda}\big]\big[(q-r)^\kappa \eta_{\lambda'\mu'} + (r-p)_{\lambda'} \eta_{\mu'}{}^\kappa + (p-q)_{\mu'} \eta^\kappa{}_{\lambda'}\big] \times$$
$$\times \Big[-\eta^{\lambda\lambda'} + (1-c)\frac{q^\lambda q^{\lambda'}}{q}\Big]\Big[-\eta^{\mu\mu'} + (1-c)\frac{r^\mu r^{\mu'}}{r}\Big]$$
$$+ 2q\cdot r \Big\}\Big|_\infty \qquad \ldots(G33)$$
$$= -\tfrac{1}{2} i g^2 C_A \int \frac{d^{2\ell}q\, d^{2\ell}r}{q^2 r^2}\, \delta(p+q+r) \times$$
$$\times \Big\{ \big[(q-r)_\kappa \eta_{\lambda\mu} + (r-p)_\lambda \eta_{\mu\kappa} + (p-q)_\mu \eta_{\kappa\lambda}\big] \times$$
$$\times \Big[(q-r)^\kappa \eta^{\lambda\mu} + (r-p)^\lambda \eta^{\mu\kappa} + (p-q)^\mu \eta^{\kappa\lambda}$$
$$-2\frac{(1-c)}{q^2}\left(p^\mu q^\lambda q^\kappa - q^\mu q^\lambda r^\kappa + q\cdot(r-p) q^\lambda \eta^{\mu\kappa}\right)$$
$$+ \frac{(1-c)^2}{q^2 r^2}\left((r\cdot p) q^\lambda r^\mu q^\kappa - (q\cdot p) q^\lambda r^\mu r^\kappa\right)\Big]$$
$$+ 2q\cdot r \Big\}\Big|_\infty$$
$$= -\tfrac{1}{2} i g^2 C_A \int \frac{d^{2\ell}q}{q^4}\left[1 - \frac{2(q\cdot p)}{q^2} - \frac{p^2}{q^2} + \frac{4(q\cdot p)^2}{q^4} + \ldots\right] \times$$
$$\times \Big\{ 6(2\ell-1)(q^2 + q\cdot p + p^2) - 2q^2 + 2q\cdot p$$
$$-2\frac{(1-c)}{q^2}\left[3q^4 + 12q^2(q\cdot p)^2 + 2q^2p^2 + 10(q\cdot p)^2\right]$$
$$+ \frac{(1-c)^2}{q^2(q+p)^2}\left[q^2p^4 - p^2(q\cdot p)^2\right]\Big\}\Big|_\infty$$

The coefficient of $(1-c)^2$ is already convergent, as is to be expected.

$$\text{So } \Pi_\kappa{}^\kappa(p)\Big|_\infty = -\tfrac{1}{2} i g^2 C_A \int \frac{d^{2\ell}q}{q^4}\left\{-2p^2 - 32\frac{(q\cdot p)^2}{q^2} + (1-c)\left[-11p^2 + 32\frac{(q\cdot p)^2}{q^2}\right]\right\}$$
$$= -\tfrac{1}{2} i g^2 C_A \left[(13-3c)p^2\right]\int \frac{d^{2\ell}q}{q^4}$$
$$= -\tfrac{1}{2} i g^2 C_A (13-3c) p^2 (2\pi)^{-4} I_1$$
$$= \tfrac{1}{2}(13-3c) p^2 C_A L \qquad \ldots(G34)$$

Whence, using (G31),

$$\Pi_c = -\frac{(13-3c)}{6} C_A L \qquad \ldots(G35)$$

SCALAR FIELD CONTRIBUTION TO THE VECTOR MESON SELF-ENERGY:

This and the following result are gauge-independent.

$$\tilde{\Pi}^{aa'}_{\kappa\kappa'}(p)\Big|_\infty = \tilde{\Pi}^{aa'}_{\kappa\kappa'(i)}(p)\Big|_\infty$$
$$= -ig^2 T_\phi \delta^{aa'} \int \frac{d^{2\ell}k}{(k^2-m^2)[(p+k)^2-m^2]} (2k+p)_\kappa (2k+p)_{\kappa'}\Big|_\infty \qquad \text{...(G36)}$$

$$= -ig^2 T_\phi \delta^{aa'} \int \frac{d^{2\ell}k}{(k^2-m^2)^2}\left[1 - \frac{2k.p}{k^2-m^2} - \frac{p^2}{k^2-m^2} + \frac{4(k.p)^2}{(k^2-m^2)^2}+\cdots\right]\times$$
$$\times\left[p_\kappa p_{\kappa'} + 2p_\kappa k_{\kappa'} + 2p_{\kappa'}k_\kappa + 4k_\kappa k_{\kappa'}\right]\Big|_\infty$$
$$= -ig^2 T_\phi \delta^{aa'} \int \frac{d^{2\ell}k}{(k^2-m^2)^2}\Big\{p_\kappa p_{\kappa'} + 4k_\kappa k_{\kappa'}\Big[1 - \frac{p^2}{k^2-m^2} + \frac{4(k.p)^2}{(k^2-m^2)^2}\Big]$$
$$+ 4k_\kappa p_{\kappa'}\Big[\frac{-2k.p}{k^2-m^2}\Big]\Big\}$$

As expected, the use of (F1) now leads to a transverse result.

$$\text{i.e. } \tilde{\Pi}^{aa'}_{\kappa\kappa'}(p)\Big|_\infty = -ig^2 T_\phi \delta^{aa'} \int \frac{d^{2\ell}k}{k^4}\left[p_\kappa p_{\kappa'} + \tfrac{2}{3}(p^2\eta_{\kappa\kappa'} + 2p_\kappa p_{\kappa'}) - \eta_{\kappa\kappa'}p^2 - 2p_\kappa p_{\kappa'}\right]$$
$$= -\tfrac{1}{3} i g^2 T_\phi \delta^{aa'}[-\eta_{\kappa\kappa'}p^2 + p_\kappa p_{\kappa'}](2\pi)^{-4} I_1$$
$$= [-\eta_{\kappa\kappa'}p^2 + p_\kappa p_{\kappa'}]\tfrac{1}{3} T_\phi \delta^{aa'} L \qquad \text{...(G37)}$$

FERMION FIELD CONTRIBUTION TO THE VECTOR MESON SELF-ENERGY:

$$\Sigma^{aa'}_{\kappa\kappa'}(p)\Big|_\infty = i g^2 T_\psi \delta^{aa'} \int \frac{d^{2\ell}k}{(k^2-m^2)[(p+k)^2-m^2]} \operatorname{Tr}[\gamma_\kappa(\not{k}+m)\gamma_{\kappa'}(\not{k}+\not{p}-m)]\Big|_\infty \qquad \text{...(G38)}$$

$$= 4ig^2 T_\psi \delta^{aa'} \int \frac{d^{2\ell}k}{(k^2-m^2)[(p+k)^2-m^2]}\left[k_\kappa(p+k)_{\kappa'} - k.(p+k)\eta_{\kappa\kappa'}\right.$$
$$\left. + (p+k)_\kappa k_{\kappa'} + m^2\eta_{\kappa\kappa'}\right]\Big|_\infty$$
$$= 4ig^2 T_\psi \delta^{aa'} \int \frac{d^{2\ell}k}{(k^2-m^2)^2}\left[1 - \frac{2k.p}{k^2-m^2} - \frac{p^2}{k^2-m^2} + \frac{4(k.p)^2}{(k^2-m^2)^2}+\cdots\right]\times$$
$$\times\left[2k_\kappa k_{\kappa'} - k^2\eta_{\kappa\kappa'} + k_\kappa p_{\kappa'} + p_\kappa k_{\kappa'} - (p.k)\eta_{\kappa\kappa'} + m^2\eta_{\kappa\kappa'}\right]\Big|_\infty$$

where (E16) has been used.

Now we put m=0 to extract the transverse divergent part.

$$\text{i.e}\ \Sigma^{aa'}_{\kappa\kappa'}(p)\Big|_\infty = 4ig^2 T_\psi \delta^{aa'} \int \frac{d^4k}{k^4} \Big\{ 2k_\kappa k_{\kappa'} \Big[1 - \frac{p^2}{k^2} + \frac{4(k.p)^2}{k^4} \Big] - 4k_\kappa p_{\kappa'} \frac{(k.p)}{k^2} + \eta_{\kappa\kappa'} \Big[p^2 - 2\frac{(k.p)^2}{k^2} \Big] \Big\}$$

$$= 4ig^2 T_\psi \delta^{aa'} \int \frac{d^4k}{k^4} \Big[\tfrac{1}{3}\eta_{\kappa\kappa'} p^2 - \tfrac{1}{3} p_\kappa p_{\kappa'} \Big]$$

$$= \frac{-4}{3}\Big[-\eta_{\kappa\kappa'} p^2 + p_\kappa p_{\kappa'} \Big] i g^2 T_\psi \delta^{aa'} (2\pi)^{-4} I_1$$

$$= \Big[-\eta_{\kappa\kappa'} p^2 + p_\kappa p_{\kappa'} \Big] \frac{4}{3} T_\psi \delta^{aa'} L \qquad \text{...(G39)}$$

SCALAR VERTEX CORRECTION:

We work at zero momentum transfer, $p=p'$, to extract the divergent part. The result is proportional to $2p_\lambda$.

$$\Gamma^a_{\lambda(i)}(p)\Big|_{\infty\,(p=p')} = i g^2 T^b T^a T^b \int \frac{d^4k}{k^2[(p+k)^2-m^2]} (2p+k)_\mu (2p+2k)_\lambda (2p+k)_\nu \Big[-\eta^{\mu\nu} + (1-c)\frac{k^\mu k^\nu}{k^2} \Big]\Big|_\infty \qquad \text{...(G40)}$$

$$= i g^2 T^b T^a T^b \int \frac{d^4k}{k^6} \Big[1 - \frac{4(k.p)}{k} + \cdots \Big] (2p+2k)_\lambda \times \Big\{ -k^2 - 4k.p - 4p^2 + (1-c)\Big[k^2 + 4k.p + \frac{4(k.p)^2}{k^2} \Big] \Big\}\Big|_\infty$$

leading to

$$\Gamma^a_{\lambda(i)}(p)\Big|_\infty = 2ig^2 T^b T^a T^b \int \frac{d^4k}{k^6} \Big[-c\, p_\lambda k^2 \Big]$$

$$= -2c\, p_\lambda\, i g^2 T^b T^a T^b (2\pi)^{-4} I_1$$

$$= (2p_\lambda)\, c\, T^b T^a T^b L \qquad \text{...(G41)}$$

$$\Gamma^a_{\lambda(ii)}(p)\Big|_{\infty\,(p=p')} = -ig^2 i f^{abc} T^b T^c \int \frac{d^4k}{(k^2-m^2)(p-k)^4} (p+k)_\mu (p+k)_\nu \times \Big[-(2p-2k)_\lambda \eta_{\rho\sigma} + (p-k)_\rho \eta_{\sigma\lambda} + (p-k)_\sigma \eta_{\lambda\rho} \Big] \times \Big[-\eta^{\mu\sigma} + (1-c)\frac{(p-k)^\mu (p-k)^\sigma}{(p-k)^2} \Big]\Big[-\eta^{\rho\nu} + (1-c)\frac{(p-k)^\rho (p-k)^\nu}{(p-k)^2} \Big]\Big|_\infty \qquad \text{...(G42)}$$

$$= -i g^2 i f^{abc} T^b T^c \int \frac{d^d k}{k^4 [(p-k)^2 - m^2]} \times$$

$$\times \{ (4k.p - 8p^2) k_\lambda + (-4k^2 + 8k.p) p_\lambda$$

$$+ 2(1-c) [(-2k.p + 4\frac{(k.p)^2}{k^2}) k_\lambda + (2k^2 - 4k.p) p_\lambda] \}\Big|_\infty$$

$$= -i g^2 i f^{abc} T^b T^c \int \frac{d^d k}{k^6} [1 + \frac{2k.p}{k} + \cdots] \times$$

$$\times \{ k_\lambda [4c(k.p) - 8p^2 + 8(1-c) \frac{(k.p)^2}{k^2}]$$

$$+ p_\lambda [-4ck^2 + 8c(k.p)] \}\Big|_\infty$$

where the shift $k \to p-k$ has been made above.

So
$$\Gamma^a_{\lambda(ii)}(p)\Big|_\infty = -4c\, i g^2 i f^{abc} T^b T^c \int \frac{d^d k}{k^6} [k_\lambda (k.p) - p_\lambda k^2]$$

$$= 3c\, p_\lambda\, i g^2 i f^{abc} T^b T^c (2\pi)^{-4} I_1$$

$$= (2p_\lambda)(-\tfrac{3}{2}c)\, i f^{abc} T^b T^c L \qquad \ldots (G43)$$

$$\Gamma^a_{\substack{\lambda(iv)+(v) \\ (p=p')}}(p)\Big|_\infty = -2ig^2 (d^{abc} T^b T^c + \tfrac{1}{n} T^a) \int \frac{d^d k}{k^2 [(p+k)^2 - m^2]} (2p+k)^\mu [-\eta_{\lambda\mu} + (1-c)\frac{k_\lambda k_\mu}{k^2}]\Big|_\infty$$

$$\ldots (G44)$$

$$= -2ig^2 (d^{abc} T^b T^c + \tfrac{1}{n} T^a) \int \frac{d^d k}{k^4} [1 - \frac{2k.p}{k^2} + \cdots] \times$$

$$\times [-2p_\lambda - c k_\lambda + (1-c) \frac{2k.p}{k^2} k_\lambda]\Big|_\infty$$

leading to

$$\Gamma^a_{\lambda(iv)+(v)}(p)\Big|_\infty = -2ig^2 (d^{abc} T^b T^c + \tfrac{1}{n} T^a) \int \frac{d^d k}{k^4} [-2p_\lambda + 2\frac{k.p}{k^2} k_\lambda]$$

$$= 3p_\lambda\, i g^2 (d^{abc} T^b T^c + \tfrac{1}{n} T^a)(2\pi)^{-4} I_1$$

$$= (2p_\lambda)(\tfrac{-3}{2})(d^{abc} T^b T^c + \tfrac{1}{n} T^a)\, L \qquad \ldots (G45)$$

Adding (G41), (G43) and (G45) gives

$$\Gamma^a_{\substack{\lambda \\ (total)}}(p)\Big|_\infty = (2p_\lambda) [c T^b T^a T^b - \tfrac{3}{2} c\, i f^{abc} T^b T^c - \tfrac{3}{2}(d^{abc} T^b T^c + \tfrac{1}{n} T^a)] L$$

$$\ldots (G46a)$$

$$= (2p_\lambda) [\frac{3(c-1)n^2 - 2c + 6}{4n}] T^a L \qquad \ldots (G46b)$$

where (B16), (B17) and (B18) have been used.

FERMION VERTEX CORRECTION:

This time when working at zero momentum transfer the result is proportional to γ_λ.

$$\Gamma^a_{\lambda(i)}(p)\Big|_{\substack{\infty \\ (p=p')}} = i g^2 T^b T^a T^b \int \frac{d^4k}{k^2[(p+k)^2-m^2]^2}\left[-\eta^{\mu\nu}+(1-c)\frac{k^\mu k^\nu}{k^2}\right] \times$$
$$\times\, \gamma_\mu\,(\not{p}+\not{k}+m)\,\gamma_\lambda(\not{p}+\not{k}+m)\gamma_\nu\Big|_\infty \qquad \text{...(G47)}$$

Only leading order in k will contribute to the divergent part.

$$\text{So}\quad \Gamma^a_{\lambda(i)}(p)\Big|_\infty = i g^2 T^b T^a T^b \int \frac{d^4k}{k^6}\left[-\eta^{\mu\nu}+(1-c)\frac{k^\mu k^\nu}{k^2}\right]\gamma_\mu \not{k}\,\gamma_\lambda \not{k}\,\gamma_\nu$$
$$= i g^2 T^b T^a T^b \int \frac{d^4k}{k^6}\left[4k_\lambda \not{k} - (1+c)k^2\gamma_\lambda\right]$$
$$= i g^2 T^b T^a T^b\,(-c\,\gamma_\lambda)\int \frac{d^4k}{k^4}$$
$$= i g^2 T^b T^a T^b\,\gamma_\lambda(-c)\,(2\pi)^{-4} I_1$$
$$= \gamma_\lambda\, c\, T^b T^a T^b\, L \qquad \text{...(G48)}$$

where (E28) and (E17) have been used.

$$\Gamma^a_{\lambda(i)}(p)\Big|_{\substack{\infty \\ (p=p')}} = -i g^2\, i f^{abc} T^b T^c \int \frac{d^4k}{(k^2-m^2)(p-k)^4}\,\gamma_\mu(\not{k}+m)\gamma_\nu \times$$
$$\times\left[-(2p-2k)_\lambda\,\eta_{\rho\sigma}+(p-k)_\rho\,\eta_{\sigma\lambda}+(p-k)_\sigma\,\eta_{\lambda\rho}\right]\times$$
$$\times\left[-\eta^{\mu\sigma}+(1-c)\frac{(p-k)^\mu(p-k)^\sigma}{(p-k)^2}\right]\left[-\eta^{\rho\nu}+(1-c)\frac{(p-k)^\rho(p-k)^\nu}{(p-k)^2}\right]\Big|_\infty$$
$$\text{...(G49)}$$

Once again only leading order in k is retained.

$$\text{So}\quad \Gamma^a_{\lambda(i)}(p)\Big|_\infty = -i g^2\, i f^{abc} T^b T^c \int \frac{d^4k}{k^6}\,\gamma_\mu \not{k}\,\gamma_\nu \times$$
$$\times\left[2k_\lambda\eta_{\rho\sigma}-k_\rho\eta_{\sigma\lambda}-k_\sigma\eta_{\lambda\rho}\right]\left[-\eta^{\mu\sigma}+(1-c)\frac{k^\mu k^\sigma}{k^2}\right]\left[-\eta^{\rho\nu}+(1-c)\frac{k^\rho k^\nu}{k^2}\right]$$
$$= -i g^2\, i f^{abc} T^b T^c \int \frac{d^4k}{k^6}\,\gamma_\mu \not{k}\,\gamma_\nu \times$$
$$\times\left\{\left[2k_\lambda\eta^{\mu\nu}-k^\nu\eta^\mu{}_\lambda-k^\mu\eta_\lambda{}^\nu\right]-\frac{(1-c)}{k^2}\left[2k_\lambda k^\mu k^\nu-k^2\eta^\mu_\lambda k^\nu-k^2\eta^\nu_\lambda k^\mu\right]\right\}$$

Using (E19) and (E17) gives:

$$\Gamma^{a}_{\lambda(1)}(p)\Big|_{\infty} = -ig^2 if^{abc}T^bT^c \int \frac{d^4k}{k^2}\left[(-6+2c)k\,k_\lambda - 2ck^2\gamma_\lambda\right]$$

$$= ig^2 if^{abc}T^bT^c\left[\tfrac{3}{2}(1+c)\gamma_\lambda\right](2\pi)^4 I_1$$

$$= \gamma_\lambda \frac{-3(1+c)}{2} if^{abc}T^bT^c L \qquad \text{...(G50)}$$

Adding (G48) and (G50) gives:

$$\Gamma^{a}_{\lambda\,(\text{total})}(p)\Big|_{\infty} = \gamma_\lambda\left[cT^bT^aT^b - \frac{3(1+c)}{2} if^{abc}T^bT^c\right]L \qquad \text{...(G51a)}$$

$$= \gamma_\lambda\left[\frac{3(1+c)n^2-2c}{4n}\right]T^aL \qquad \text{...(G51b)}$$

where (B16) and (B17) have been used.

GHOST SELF-ENERGY:

$$\Pi^{aa'}(p^2)\Big|_{\infty} = -ig^2 C_A\delta^{aa'}\int \frac{d^4k}{k^2(p+k)^2}(p+k)_\mu p_\nu\left[-\eta^{\mu\nu}+(1-c)\frac{k^\mu k^\nu}{k^2}\right]\Big|_{\infty} \qquad \text{...(G52)}$$

$$= -ig^2 C_A\delta^{aa'}\int \frac{d^4k}{k^4}\left[1-\frac{2k.p}{k}+\dots\right]\times$$

$$\times\left[-p^2-(k.p)+(1-c)\left(\frac{(k.p)^2}{k^2}+(k.p)\right)\right]\Big|_{\infty}$$

leading to

$$\Pi^{aa'}(p^2)\Big|_{\infty} = -ig^2 C_A\delta^{aa'}\int\frac{d^4k}{k^4}\left[-p^2+(1+c)\frac{(k.p)^2}{k^2}\right]$$

$$= -ig^2 C_A\delta^{aa'}\left[-p^2+\frac{(1+c)}{4}p^2\right]\int\frac{d^4k}{k^4}$$

$$= ig^2 C_A\delta^{aa'}\frac{(3-c)}{4}p^2(2\pi)^4 I_1$$

$$= p^2\frac{(c-3)}{4}C_A\delta^{aa'}L \qquad \text{...(G53)}$$

GHOST VERTEX CORRECTION:

We work at zero momentum transfer to extract the divergent part. The result is proportional to p_λ since the vertex is one-sided.

$$\Gamma^{abc}_{\lambda(i)\,(p=p')}(p)\Big|_{\infty} = -\tfrac{1}{2}g^2 C_A f^{abc}\int\frac{d^4k}{k^2(p+k)^4}(p+k)_\lambda(p+k)_\mu p_\nu\left[-\eta^{\mu\nu}+(1-c)\frac{k^\mu k^\nu}{k^2}\right]\Big|_{\infty}$$

...(G54)

Here, and in the following, only leading order in k is retained for the divergent part.

So $$\Gamma^{abc}_{\lambda(i)}(p)\Big|_\infty = -\tfrac{1}{2} g^2 C_A f^{abc} \int \frac{d^\nu k}{k^6}\, k_\lambda k_\mu p_\nu \left[-\eta^{\mu\nu} + (1-c)\frac{k^\mu k^\nu}{k^2}\right]$$
$$= +\tfrac{1}{2} g^2 C_A f^{abc} c \int \frac{d^\nu k}{k^6} (k.p) k_\lambda$$
$$= +\tfrac{1}{8} g^2 C_A f^{abc} c\, p_\lambda (2\pi)^{-4} I_1$$
$$= (i p_\lambda)\left(\tfrac{+c}{8}\right) C_A f^{abc} L \qquad \ldots \text{(G55)}$$

$$\Gamma^{abc}_{\lambda(ii)}(p)\Big|_{\infty\,(p=p')} = -\tfrac{1}{2} g^2 C_A f^{abc} \int \frac{d^\nu k}{k^2 (p-k)^4}\, k_\mu p_\nu \times$$
$$\times \left[(p-k)_\sigma \eta_{\lambda\rho} + 2(k-p)_\lambda \eta_{\rho\sigma} + (p-k)_\rho \eta_{\sigma\lambda}\right] \times$$
$$\times \left[-\eta^{\mu\sigma} + (1-c)\frac{(p-k)^\mu (p-k)^\sigma}{(p-k)^2}\right]\left[-\eta^{\nu\rho} + (1-c)\frac{(p-k)^\nu (p-k)^\rho}{(p-k)^2}\right]\Big|_\infty \qquad \ldots \text{(G56)}$$

leading to

$$\Gamma^{abc}_{\lambda(ii)}(p)\Big|_\infty = -\tfrac{1}{2} g^2 C_A f^{abc} \int \frac{d^\nu k}{k^6}\, k_\mu p_\nu \left[-k_\sigma \eta_{\lambda\rho} + 2k_\lambda \eta_{\rho\sigma} - k_\rho \eta_{\sigma\lambda}\right] \times$$
$$\times \left[-\eta^{\mu\sigma} + (1-c)\frac{k^\mu k^\sigma}{k^2}\right]\left[-\eta^{\nu\rho} + (1-c)\frac{k^\nu k^\rho}{k^2}\right]$$
$$= -\tfrac{1}{2} g^2 C_A f^{abc} \int \frac{d^\nu k}{k^6}\, k_\mu p_\nu \times$$
$$\times \left\{-k^\mu \eta_\lambda{}^\nu + 2k_\lambda \eta^{\mu\nu} - k^\nu \eta_\lambda{}^\mu - \frac{(1-c)}{k^2}\left[-k^2 k^\mu \eta_\lambda{}^\nu - k^2 k^\nu \eta_\lambda{}^\mu + 2k^\mu k^\nu k_\lambda\right]\right\}$$
$$= -\tfrac{1}{2} g^2 C_A f^{abc} c \int \frac{d^\nu k}{k^6}\left[-k^2 p_\lambda + (k.p) k_\lambda\right]$$
$$= \tfrac{3}{8} g^2 C_A f^{abc} c\, p_\lambda (2\pi)^{-4} I_1$$
$$= (i p_\lambda) \frac{3c}{8} C_A f^{abc} L \qquad \ldots \text{(G57)}$$

Adding (G55) and (G57) gives

$$\Gamma^{abc}_{\lambda\,(total)}(p)\Big|_\infty = (i p_\lambda) \frac{c}{2} C_A f^{abc} L \qquad \ldots \text{(G58)}$$

AXIAL GAUGE

SCALAR MESON SELF-ENERGY

$$\Pi(p^2)\Big|_\infty = \Pi_{vi}(p^2)\Big|_\infty$$
$$= -i g^2 C_\phi \int \frac{d^{2\ell}k}{k^2[(p+k)^2-m^2]} (2p+k)_i (2p+k)_j \left[\delta_{ij} - \frac{k_i k_j}{k_0^2}\right]\Big|_\infty \qquad \text{...(G59)}$$

$$= -i g^2 C_\phi \int \frac{d^{2\ell}k}{k^4}\left[1 - \frac{2k.p}{k^2} - \frac{p^2-m^2}{k^2} + \frac{4(k.p)^2}{k^4} + \cdots\right] \times$$
$$\times \left[\underline{k}^2 + 4\underline{k}.\underline{p} + 4\underline{p}^2 - \frac{1}{k_0^2}\left(\underline{k}^4 + 4\underline{k}^2(\underline{k}.\underline{p}) + 4(\underline{k}.\underline{p})^2\right)\right]\Big|_\infty$$

leading to

$$\Pi(p^2)\Big|_\infty = -i g^2 C_\phi \int \frac{d^{2\ell}k}{k^4}\Big\{-(p^2-m^2)\frac{\underline{k}^2}{k^2} + 4\frac{\underline{k}^2}{k^2}(k_0^2p_0^2 + (\underline{k}.\underline{p})^2) + 8\frac{(\underline{k}.\underline{p})^2}{k^2} + 4\underline{p}^2$$
$$- \frac{1}{k_0^2}\left[-(p^2-m^2)\frac{\underline{k}^4}{k^2} + 4\frac{\underline{k}^4}{k^4}(k_0^2p_0^2 + (\underline{k}.\underline{p})^2) + 8\underline{k}^2\frac{(\underline{k}.\underline{p})^2}{k^2} + 4(\underline{k}.\underline{p})^2\right]\Big\}$$
$$= -i g^2 C_\phi \int d^{2\ell}k\Big[\left(-p^2 + m^2 + \tfrac{4}{3}\underline{p}^2\right)\frac{\underline{k}^2}{k_0^2 k^4} + 4p_0^2\frac{\underline{k}^2}{k^6} + 4\underline{p}^2\frac{1}{k^4}$$
$$+ \frac{4}{3}\underline{p}^2\left(\frac{\underline{k}^4}{k^8} - \frac{\underline{k}^6}{k_0^2 k^8}\right)\Big]$$
$$= -i g C_\phi (2\pi)^{-4}\Big[\left(-p^2 + m^2 + \tfrac{4}{3}\underline{p}^2\right) I_8 + 4p_0^2 I_2 + 4\underline{p}^2 I_1$$
$$+ \frac{4}{3}\underline{p}^2 (I_4 - I_{10})\Big]$$
$$= (-6p^2 + 3m^2)\, C_\phi\, L \qquad \text{...(G60)}$$

FERMION SELF-ENERGY:

$$\Sigma(p)\Big|_\infty = -i g^2 C_\psi \int \frac{d^{2\ell}k}{k^2[(p+k)^2-m^2]} \gamma_i (\not p + \not k + m)\gamma_j \left[\delta_{ij} - \frac{k_i k_j}{k_0^2}\right]\Big|_\infty \qquad \text{...(G61)}$$

$$= -i g^2 C_\psi \int \frac{d^{2\ell}k}{k^4}\left[1 - \frac{2k.p}{k} + \cdots\right] \times$$
$$\times \Big\{3\gamma_0 k_0 - \underline{\gamma}.\underline{k} + 3\gamma_0 p_0 - \underline{\gamma}.\underline{p}$$
$$- \frac{1}{k_0^2}\left[\underline{k}^2\gamma_0 k_0 + \underline{k}^2\underline{\gamma}.\underline{k} + 2\underline{k}.\underline{p}\,\underline{\gamma}.\underline{k} - \underline{k}^2\underline{\gamma}.\underline{p} + \underline{k}^2\gamma_0 p_0\right]$$
$$- 3m^2 + m^2\frac{\underline{k}^2}{k_0^2}\Big\}\Big|_\infty$$

where (E23), (E24), (E5) and (E18) have been used.

$$\text{So}\quad \Sigma(p)\big|_\infty = -ig^2 C_\gamma \int \frac{d^\ell k}{k^4}\Big\{ -6\gamma_0 p_0 \frac{k_0^2}{k^2} - \frac{2}{3}\underline{\gamma}\cdot\underline{p}\,\frac{\underline{k}^2}{k^2} + 3\gamma_0 p_0 - \underline{\gamma}\cdot\underline{p}$$

$$- \frac{1}{k_0^2}\Big[-2\gamma_0 p_0 \frac{\underline{k}^2 k_0^2}{k^2} + \frac{2}{3}\underline{\gamma}\cdot\underline{p}\,\frac{\underline{k}^4}{k^2} - \frac{1}{3}\underline{\gamma}\cdot\underline{p}\,\underline{k}^2 + \gamma_0 p_0 \underline{k}^2\Big] - 3m^2 + m^2\frac{\underline{k}^2}{k_0^2}\Big\}$$

$$= -ig^2 C_\gamma (2\pi)^{-4}\Big[-6\gamma_0 p_0 I_3 + \big(-\tfrac{2}{3}\underline{\gamma}\cdot\underline{p} + 2\gamma_0 p_0\big) I_2 + \big(3\gamma_0 p_0 - \underline{\gamma}\cdot\underline{p} - 3m^2\big) I$$

$$- \frac{2}{3}\underline{\gamma}\cdot\underline{p}\, I_9 + \big(\tfrac{1}{3}\underline{\gamma}\cdot\underline{p} - \gamma_0 p_0 + m^2\big) I_8\Big]$$

$$= -3\not{p}\, C_\gamma L \qquad \ldots\text{(G62)}$$

VECTOR MESON SELF-ENERGY:

Due to the absence of the ghost field in this gauge and the vanishing of seagulls, there is only one diagram contributing.

$$\text{i.e.}\quad \Pi^{aa'}_{kk'}(p)\big|_\infty = \frac{1}{2}\Pi^{aa'}_{kk'(i)}(p)\big|_\infty$$

$$= -\frac{1}{2} i g^2 C_A \delta^{aa'} \int \frac{d^{2\ell}q\, d^{2\ell}r}{q^2 r^2}\,\delta(p+q+r)\,\times$$

$$\times\big[-(q-r)_k \delta_{\ell m} + (r-p)_\ell \eta_{mk} + (p-q)_m \eta_{k\ell}\big]\big[-(q-r)_{k'}\delta_{\ell' m'} + (r-p)_{\ell'}\eta_{m'k'} + (p-q)_{m'}\eta_{k'\ell'}\big]\times$$

$$\times\Big[\delta_{\ell\ell'} - \frac{q_\ell q_{\ell'}}{q_0^2}\Big]\Big[\delta_{mm'} - \frac{r_m r_{m'}}{r_0^2}\Big]\Big|_\infty$$

$$\ldots\text{(G63)}$$

First we show that $\Pi^{aa'}_{kk'}(p)\big|_\infty$ is doubly transverse:

$$p^k p^{k'} \Pi^{aa'}_{kk'}(p)\big|_\infty = -\frac{1}{2} i g^2 C_A \delta^{aa'} \int \frac{d^{2\ell}q\, d^{2\ell}r}{q^2 r^2}\,\delta(p+q+r)\,\times$$

$$\times\big[(r^2-q^2)\delta_{\ell m} - q_\ell q_m + r_\ell r_m\big]\big[(r^2-q^2)\delta_{\ell' m'} - q_{\ell'} q_{m'} + r_{\ell'} r_{m'}\big]\times$$

$$\times\Big[\delta_{\ell\ell'} - \frac{q_\ell q_{\ell'}}{q_0^2}\Big]\Big[\delta_{mm'} - \frac{r_m r_{m'}}{r_0^2}\Big]\Big|_\infty \qquad \ldots\text{(G64)}$$

where (F10) has been used.

$$p^\kappa p^{\kappa'} \Pi^{aa'}_{\kappa\kappa'}(p)\Big|_\infty = -\tfrac{1}{2} i g^2 C_A \delta^{aa'} \int \frac{d^{2l}q\, d^{2l}r}{q^2\, r^2} \delta(p+q+r) \times$$
$$\times \Big\{ (r^2-q^2)\delta_{lm} - q_l q_m + r_l r_m - \tfrac{1}{q_0^2}\big[(r^2-q_0^2)q_m + \underline{q}\cdot\underline{r}\, r_m\big] q_l \Big\} \times$$
$$\times \Big\{ (r^2-q^2)\delta_{lm} - q_l q_m + r_l r_m - \tfrac{1}{r_0^2}\big[(r_0^2-q^2) r_l - \underline{r}\cdot\underline{q}\, q_l\big] r_m \Big\}\Big|_\infty$$
$$= -\frac{1}{2} i g^2 C_A \delta^{aa'} \int \frac{d^{2l}q\, d^{2l}r}{q^2\, r^2} \delta(p+q+r) \times$$
$$\times \Big\{ 3(r^2-q^2)^2\Big|_{(*)} + 2(r^2-q^2)(\underline{r}^2-\underline{q}^2)\Big|_{(+)} + 2\underline{q}^4 - 2(\underline{q}\cdot\underline{r})^2$$
$$- \frac{2}{q_0^2}\big[(r^2-q_0^2)q_m + \underline{q}\cdot\underline{r}\, r_m\big]\big[(r^2-q_0^2)q_m + \underline{q}\cdot\underline{r}\, r_m\big]$$
$$+ \frac{1}{q_0^2 r_0^2} (\underline{r}\cdot\underline{q})^2 (r_0^2 - q_0^2)^2 \Big\}\Big|_\infty$$

The indicated terms are now shown to give zero divergent contribution:

$$(*): \quad \int \frac{d^{2l}q\, d^{2l}r}{q^2\, r^2} \delta(p+q+r)\big[r^4 - 2q^2 r^2 + q^4\big]\Big|_\infty$$
$$= \quad 2\int \frac{d^{2l}q\, r^2}{q^2} - 2\int d^{2l}q\ \Big|_\infty$$
$$= \quad 2\int d^{2l}q \Big[2\frac{q\cdot p}{q^2} + \frac{p^2}{q^2}\Big]\Big|_\infty$$
$$= \quad 0 \qquad \ldots\text{(G65)}$$

$$(+): \quad \int \frac{d^{2l}q\, d^{2l}r}{q^2\, r^2} \delta(p+q+r)(r^2\underline{r}^2 - q^2\underline{r}^2 - \underline{q}^2 r^2 + q^2\underline{q}^2)\Big|_\infty$$
$$= \quad 2\int \frac{d^{2l}q\, \underline{r}^2}{q^2} - 2\int \frac{d^{2l}q\, \underline{q}^2}{q^2}\ \Big|_\infty$$
$$= \quad 2\int \frac{d^{2l}q}{q^2}(2\underline{q}\cdot\underline{p} + \underline{p}^2)\Big|_\infty$$
$$= \quad 0 \qquad \ldots\text{(G66)}$$

So we are left with a straightforward, if complicated, calculation:

$$p^\kappa p^{\kappa'} \Pi^{aa'}_{\kappa\kappa'}(p)\Big|_\infty = -\tfrac{1}{2} i g^2 C_A \delta^{aa'} \int \frac{d^{2\ell}q\, d^{2\ell}r}{q^2 r^2} \delta(p+q+r) \times$$
$$\times \Big\{ 2\underline{q}^4 - 2(\underline{q}\cdot\underline{r})^2$$
$$- \frac{2}{q_0^2}\Big[r^4 \underline{q}^2 - 2r^2 q_0^2 \underline{q}^2 + q_0^4 \underline{q}^2 + 2r^2(\underline{q}\cdot\underline{r})^2 - 2q_0^2(\underline{q}\cdot\underline{r})^2 + \underline{r}^2(\underline{q}\cdot\underline{r})^2\Big]$$
$$+ 2\frac{(\underline{q}\cdot\underline{r})^2}{q_0^2 r_0^2}\Big[q_0^4 - q_0^2 r_0^2\Big] \Big\}\Big|_\infty$$

Thus, after cancellations,

$$p^\kappa p^{\kappa'} \Pi^{aa'}_{\kappa\kappa'}(p)\Big|_\infty = i g^2 C_A \delta^{aa'} \int d^{2\ell}q\, d^{2\ell}r\, \delta(p+q+r) \times$$
$$\times \Big\{ \frac{\underline{q}^2}{r^2} + \frac{1}{q^2 q_0^2}\Big[r^2 \underline{q}^2 + (\underline{q}\cdot\underline{r})^2\Big] \Big\}\Big|_\infty$$
$$= i g^2 C_A \delta^{aa'} \int d^{2\ell}q \Big\{ \frac{1}{q^2 q_0^2}\Big[(q^2 + 2q\cdot p + p^2)\underline{q}^2 + (\underline{q}^2 + \underline{q}\cdot\underline{p})^2\Big]$$
$$+ \frac{\underline{q}^2}{q^2}\Big[1 - \frac{2q\cdot p}{q^2} - \frac{p^2}{q^2} + \frac{4(q\cdot p)^2}{q^4} + \frac{4p^2(q\cdot p)}{q^6} - \frac{8(q\cdot p)^3}{q^6}$$
$$+ \frac{p^4}{q^4} - \frac{12p^2(q\cdot p)^2}{q^6} + \frac{16(q\cdot p)^4}{q^8} + \cdots\cdots \Big] \Big\}\Big|_\infty$$

The first term does not contribute. The rest gives

$$p^\kappa p^{\kappa'} \Pi^{aa'}_{\kappa\kappa'}(p)\Big|_\infty = i g^2 C_A \delta^{aa'} \int d^{2\ell}q\, \frac{\underline{q}^2}{q^2}\Big[\frac{p^4}{q^4} - 12p^2 p_0^2 \frac{q_0^2}{q^6} - 12p^2 \frac{(\underline{q}\cdot\underline{p})^2}{q^6} + 16 p_0^4 \frac{q_0^4}{q^8}$$
$$+ 96 p_0^2 \frac{q_0^2(\underline{q}\cdot\underline{p})^2}{q^8} + 16\frac{(\underline{q}\cdot\underline{p})^4}{q^8}\Big]$$
$$= i g^2 C_A \delta^{aa'} (2\pi)^{-4} \Big\{ p_0^4\big[I_2 - 12 I_5 + 16 I_8\big]$$
$$+ p_0^2 \underline{p}^2 \big[-2I_2 + 12 I_5 - 4I_4 + 32 I_{17}\big]$$
$$+ \underline{p}^4 \big[I_2 + 4I_4 + \tfrac{16}{5} I_{16}\big] \Big\}$$
$$= 0 \qquad \ldots(G67)$$

Thus $\Pi^{aa'}_{\kappa\kappa'}(p)\big|_\infty$ is doubly transverse, and as discussed in the text it takes the same form as (G31), i.e. $\Pi_{\kappa\kappa'}(p)\big|_\infty = (-\eta_{\kappa\kappa'}p^2 + p_\kappa p_{\kappa'})\,\Pi_c$ ...(G68)

So it is only necessary to evaluate

$$\Pi_{00}(p)\big|_\infty = \mathbf{p}^2\,\Pi_c \qquad \text{...(G69)}$$

where the factor $\mathbf{p}^2$ serves as a check on the calculation.

$$\Pi_{00}(p)\big|_\infty = -\tfrac{1}{2}\,i g^2 C_A \int \frac{d^{2\ell}q\, d^{2\ell}r}{q^2 r^2}\,\delta(p+q+r)\,\times$$

$$\times\,(q_0-r_0)^2\,\delta_{\ell m}\delta_{\ell' m'}\left[\delta_{\ell\ell'} - \frac{q_\ell q_{\ell'}}{q_0^2}\right]\left[\delta_{mm'} - \frac{q_m q_{m'}}{q_0^2}\right]\Big|_\infty \qquad \text{...(G70)}$$

$$= -\tfrac{1}{2}\,i g^2 C_A \int \frac{d^{2\ell}q\, d^{2\ell}r}{q^2 r^2}\,\delta(p+q+r)\,(q_0-r_0)^2\left[\,3\Big|_{(*)} - 2\frac{\mathbf{q}^2}{q_0^2}\Big|_{(+)} + \frac{(\mathbf{q}\cdot\mathbf{r})^2}{q_0^2 r_0^2}\Big|_{(*)}\right]\Big|_\infty$$

The indicated terms are each analysed separately:

$(*)$:
$$3\int d^{2\ell}q\, d^{2\ell}r\,\delta(p+q+r)\,\frac{q_0^2 - 2q_0 r_0 + r_0^2}{q^2 r^2}\Big|_\infty$$

$$= 6\int d^{2\ell}q\, d^{2\ell}r\,\delta(p+q+r)\,\frac{q_0^2 - q_0 r_0}{q^2 r^2}\Big|_\infty$$

$$= 6\int \frac{d^{2\ell}q}{q^4}\,(2q_0^2 + q_0 p_0)\left[1 - \frac{2q\cdot p}{q^2} - \frac{p^2}{q^2} + \frac{4(q\cdot p)^2}{q^4} + \cdots\right]\Big|_\infty$$

$$= 6\int \frac{d^{2\ell}q}{q^4}\left[-2p^2\frac{q_0^2}{q^2} + 8p_0^2\frac{q_0^4}{q^4} + \frac{8}{3}\mathbf{p}^2\frac{q_0^2\mathbf{q}^2}{q^4} - 2p_0^2\frac{q_0^2}{q^2}\right]$$

$$= (2\pi)^{-4}\left[(-24p_0^2 + 12\mathbf{p}^2)I_3 + 48p_0^2 I_6 + \frac{48}{3}\mathbf{p}^2 I_5\right]$$

...(G71)

$(+)$:
$$= -2\int d^{2\ell}q\, d^{2\ell}r\,\delta(p+q+r)\,\frac{\mathbf{q}^2(q_0-r_0)^2}{q_0^2 q^2 r^2}\Big|_\infty$$

$$= -2\int d^{2\ell}q\, d^{2\ell}r\,\delta(p+q+r)\left[\frac{1}{q^2 r^2} - \frac{1}{q_0^2 r^2}\right]\Big|_\infty$$

$$= -2\int d^{2\ell}q\,(4q_0^2 + 4q_0 p_0 + p_0^2)\left[1 - \frac{2q\cdot p}{q^2} + \cdots\cdots\right]\left[\frac{1}{q^4} - \frac{1}{q_0^2 q^2}\right]\Big|_\infty$$

$$= -2\int d^{2\ell}q\,\Big[-4p^2\frac{q_0^2}{q^6} + 16p_0^2\frac{q_0^4}{q^8} - 8p_0^2\frac{q_0^2}{q^6} + p_0^2\frac{1}{q^4} + \frac{16}{3}\mathbf{p}^2\frac{q_0^2\mathbf{q}^2}{q^8}$$

$$+ 4p^2\frac{1}{q^4} - 16p\,\frac{q_0^2}{q^6} - p_0^2\frac{1}{q^2 q_0^2} + 8p_0^2\frac{1}{q^4} - \frac{16}{3}\mathbf{p}^2\frac{\mathbf{q}^2}{q^6}\Big]$$

$$= (2\pi)^{-4}\Big[(56p_0^2 - 8\mathbf{p}^2)I_3 - 32p_0^2 I_6 + (-26p_0^2 + 8\mathbf{p}^2)I_1$$

$$- \frac{32}{3}\mathbf{p}^2 I_5 + 2p_0^2 I_7 + \frac{32}{3}\mathbf{p}^2 I_2\Big]$$

...(G72)

$$(\ddagger)\ :\quad \int d^nq\, d^nr\, \delta(p+q+r)\, \frac{(q_0-r_0)^2(\mathbf{q}\cdot\mathbf{r})^2}{q^2r^2q_0^2r_0^2}\Big|_\infty$$

$$= \int d^nq\, d^nr\, \delta(p+q+r)\,(q_0-r_0)^2\Big[\frac{(\mathbf{q}\cdot\mathbf{r})^2}{q^2r^2q_0^2r_0^2} - \frac{1}{q^2r^2} + \frac{2(\mathbf{q}\cdot\mathbf{r})}{q^2r^2q_0r_0}\Big]\Big|_\infty$$

$$= \int d^nq\, d^nr\, \delta(p+q+r)\Big\{\frac{2(\mathbf{q}\cdot\mathbf{r})^2}{q^2r^2r_0^2} + \Big[4q_0(\mathbf{q}\cdot\mathbf{r}) - \frac{2(\mathbf{q}\cdot\mathbf{r})^2}{q_0}\Big]\frac{1}{q^2r^2r_0}$$
$$+ \Big[-4(\mathbf{q}\cdot\mathbf{r}) + 2q_0r_0 - 2q_0^2\Big]\frac{1}{q^2r^2}\Big\}\Big|_\infty$$

$$= \int d^nq\, \Big\{\frac{2}{q^4q_0^2}\big[\mathbf{q}^4 + 2\mathbf{q}^2(\mathbf{q}\cdot\mathbf{p}) + (\mathbf{q}\cdot\mathbf{p})^2\big]\times$$
$$\times\Big[1 - \frac{2q\cdot p}{q^2} - \frac{p^2}{q^2} + \frac{4(q\cdot p)^2}{q^4} + \cdots\Big]\Big[1 - \frac{2q_0p_0}{q_0^2} - \frac{p_0^2}{q_0^2} + \frac{4q_0^2p_0^2}{q_0^4} + \cdots\Big]$$
$$+ \frac{1}{q^4}\Big[4(\mathbf{q}^2 + \mathbf{q}\cdot\mathbf{p}) + \frac{2}{q_0}\big(\mathbf{q}^4 + 2\mathbf{q}^2(\mathbf{q}\cdot\mathbf{p}) + (\mathbf{q}\cdot\mathbf{p})^2\big)\Big]\times$$
$$\times\Big[1 - \frac{2q\cdot p}{q^2} - \frac{p^2}{q^2} + \frac{4(q\cdot p)^2}{q^4} + \cdots\Big]\Big[1 - \frac{p_0}{q_0} + \frac{p_0^2}{q_0^2} + \cdots\Big]$$
$$+ \frac{1}{q^4}\Big[4\mathbf{q}^2 + 4(\mathbf{q}\cdot\mathbf{p}) - 4q_0^2 - 2q_0p_0\Big]\times$$
$$\times\Big[1 - \frac{2q\cdot p}{q^2} - \frac{p^2}{q^2} + \frac{4(q\cdot p)^2}{q^4} + \cdots\Big]\Big\}\Big|_\infty$$

$$= \int d^nq\, \Big[-4p^2\frac{1}{q^2q_0^2} + 4p_0^2\frac{1}{q^4} + 8p_0^2\frac{1}{q_0^4} + \Big(\tfrac{4}{3}\mathbf{p}^2 + 4p_0^2\Big)\frac{\mathbf{q}^2}{q^4q_0^2}$$
$$+\Big(-8p^2 + 8p_0^2 + \tfrac{16}{3}\mathbf{p}^2\Big)\frac{\mathbf{q}^2}{q^6} + \Big(32p_0^2 - \tfrac{16}{3}\mathbf{p}^2\Big)\frac{q_0^2\mathbf{q}^2}{q^8} + \frac{32}{3}\mathbf{p}^2\frac{\mathbf{q}^4}{q^8}$$
$$+ \big(4p^2 + 4p_0^2\big)\frac{q_0^2}{q^6} - 16p_0^2\frac{q_0^4}{q^8}\Big]$$

$$= (2\pi)^{-4}\Big[\big(-4p_0^2 + 4\mathbf{p}^2\big)I_7 + 4p_0^2I_1 + \big(4p_0^2 + \tfrac{4}{3}\mathbf{p}^2\big)I_8 + \tfrac{40}{3}\mathbf{p}^2I_2$$
$$+ \big(32p_0^2 - \tfrac{16}{3}\mathbf{p}^2\big)I_5 + \tfrac{32}{3}\mathbf{p}^2I_4 + \big(8p_0^2 - 4\mathbf{p}^2\big)I_3 - 16p_0^2I_6\Big] \qquad \text{...(G73)}$$

Combining (G71), (G72) and (G73) gives

$$\Pi_{00}(p)\Big|_\infty = -\tfrac{1}{2}ig^2\, C_A\,(2\pi)^{-4}\Big\{p_0^2\big[40I_3 - 22I_1 - 2I_7 + 4I_8 + 32I_5\big]$$
$$+\mathbf{p}^2\big[8I_1 + 24I_2 + 4I_7 + \tfrac{4}{3}I_8 + \tfrac{32}{3}I_4\big]\Big\}$$

$$= \mathbf{p}^2\big(-\tfrac{11}{3}\big)C_A\, L \qquad \text{...(G74)}$$

Whence, using (G69),

$$\Pi_c = -\tfrac{11}{3}C_A L \qquad \text{...(G75)}$$

SCALAR VERTEX CORRECTION:

When working at zero momentum transfer to extract the divergent parts, the results are proportional to $2p_0$ and $2p_\ell$ respectively.

$$\Gamma^a_{\lambda(i)}(p)\Big|_{\substack{\infty\\(p=p')}} = i g^2 T^b T^a T^b \int \frac{d^4k}{k^2[(p+k)^2-m^2]^2}(2p+2k)_\lambda (2p+k)_i (2p+k)_j \left[\delta_{ij} - \frac{k_i k_j}{k_0^2}\right]\Big|_\infty$$

$$= i g^2 T^b T^a T^b \int \frac{d^4k}{k^6}(2p+2k)_\lambda \left[1 - \frac{4k.p}{k^2} - \frac{2(p^2-m^2)}{k^2} + \cdots\right] \times$$

$$\times \left[(2\underline{p}+\underline{k})^2 - \frac{(\underline{k}.(2\underline{p}+\underline{k}))^2}{k_0^2}\right]\Big|_\infty \qquad \ldots (G76)$$

So $$\Gamma^a_{0(i)}(p)\Big|_\infty = i g^2 T^b T^a T^b \int \frac{d^4k}{k^6}(2p_0+2k_0)\left[1 - \frac{4k.p}{k^2} + \cdots\right] \times$$

$$\times \left[\underline{k}^2 + 4\underline{k}.\underline{p} + 4\underline{p}^2 - \frac{1}{k_0^2}\left(\underline{k}^4 + 4\underline{k}^2(\underline{k}.\underline{p}) + 4(\underline{k}.\underline{p})^2\right)\right]\Big|_\infty$$

$$= p_0\, i g^2 T^b T^a T^b \int \frac{d^4k}{k^6}\left[2\underline{k}^2 - 2\frac{\underline{k}^4}{k_0^2} - 8\frac{k_0^2\underline{k}^2}{k^2} + 8\frac{\underline{k}^4}{k^2}\right]$$

$$= p_0\, i g^2 T^b T^a T^b (2\pi)^{-4}\left[2I_2 - 2I_9 - 8I_5 + 8I_4\right]$$

$$= (2p_0)(-6)\, T^b T^a T^b L \qquad \ldots (G77)$$

and

$$\Gamma^a_{\ell(i)}(p)\Big|_\infty = i g^2 T^b T^a T^b \int \frac{d^4k}{k^6}(2p_\ell+2k_\ell)\left[1 - \frac{4k.p}{k^2} + \cdots\right] \times$$

$$\times \left[\underline{k}^2 + 4\underline{k}.\underline{p} + 4\underline{p}^2 - \frac{1}{k_0^2}\left(\underline{k}^4 + 4\underline{k}^2(\underline{k}.\underline{p}) + 4(\underline{k}.\underline{p})^2\right)\right]\Big|_\infty$$

$$= p_\ell\, i g^2 T^b T^a T^b \int \frac{d^4k}{k^6}\left[\frac{14}{3}\underline{k}^2 - \frac{14}{3}\frac{\underline{k}^4}{k_0^2} + \frac{8}{3}\frac{\underline{k}^4}{k^2} - \frac{8}{3}\frac{\underline{k}^6}{k_0^2 k^2}\right]$$

$$= p_\ell\, i g^2 T^b T^a T^b (2\pi)^{-4}\left[\frac{14}{3}I_2 - \frac{14}{3}I_9 + \frac{8}{3}I_4 - \frac{8}{3}I_{10}\right]$$

$$= (2p_\ell)(-2)\, T^b T^a T^b L \qquad \ldots (G78)$$

$$\Gamma^a_{\lambda(ii)}(p)\Big|_{\substack{\infty\\(p=p')}} = -i g^2 i f^{abc} T^b T^c \int \frac{d^4k}{(k^2-m^2)(p-k)^4}(p+k)_m (p+k)_n \times$$

$$\times \left[(2p-2k)_\lambda \delta_{rs} - (p-k)_r \delta_{s\lambda} - (p-k)_s \delta_{\lambda r}\right]\left[\delta_{ms} - \frac{(p-k)_m(p-k)_s}{(p-k)_0^2}\right]\left[\delta_{rn} - \frac{(p-k)_r(p-k)_n}{(p-k)_0^2}\right]\Big|_\infty$$

$$= -i g^2 i f^{abc} T^b T^c \int \frac{d^4k}{k^4[(p-k)^2-m^2]}\left[2k_\lambda \delta_{rs} - k_r \delta_{s\lambda} - k_s \delta_{r\lambda}\right] \times$$

$$\times \left[(2p-k)_s - \frac{\underline{k}.(2\underline{p}-\underline{k})k_s}{k_0^2}\right]\left[(2p-k)_r - \frac{\underline{k}.(2\underline{p}-\underline{k})k_r}{k_0^2}\right]\Big|_\infty \qquad \ldots (G79)$$

where the shift $k \to p-k$ has been made.

So $$\Gamma^a_{0(ii)}(p)\Big|_\infty = -ig^2 i f^{abc}T^b T^c \int \frac{d^4k}{k^6}\left[1+\frac{2k\cdot p}{k^2}+\cdots\right] 2k_0 \times\left\{\underline{k}^2 - 4\underline{k}\cdot\underline{p} + 4\underline{p}^2 - \frac{2}{k_0^2}\left[\underline{k}^4 - 4\underline{k}^2(\underline{k}\cdot\underline{p}) + 4(\underline{k}\cdot\underline{p})^2\right] + \frac{1}{k_0^4}\left[\underline{k}^6 - 4\underline{k}^4(\underline{k}\cdot\underline{p}) + 4\underline{k}^2(\underline{k}\cdot\underline{p})^2\right]\right\}\Big|_\infty$$

$$= -p_0 ig^2 i f^{abc}T^bT^c \int d^4k \left[4\frac{k_0^2\underline{k}^2}{k^8} - 8\frac{\underline{k}^4}{k^8} + 4\frac{\underline{k}^6}{k_0^2 k^8}\right]$$

$$= -p_0 ig i f^{abc}T^bT^c (2\pi)^{-4}\left[4I_5 - 8I_4 + 4I_{10}\right]$$

$$= (2p_0)\, 6 i f^{abc}T^bT^c L \qquad \ldots(G80)$$

and

$$\Gamma^a_{\ell(ii)}(p)\Big|_\infty = -ig^2 i f^{abc}T^bT^c \int \frac{d^4k}{k^6}\left[1+\frac{2k\cdot p}{k^2}+\cdots\right]\left[2k_\ell\delta_{rs} - k_r\delta_{s\ell} - k_s\delta_{\ell r}\right]\times \times\left[(2p-k)_s - \frac{\underline{k}\cdot(2\underline{p}-\underline{k})k_s}{k_0^2}\right]\left[(2p-k)_r - \frac{\underline{k}\cdot(2\underline{p}-\underline{k})k_r}{k_0^2}\right]$$

$$= -ig^2 i f^{abc}T^bT^c \int \frac{d^4k}{k^6}\left[-4k_\ell(\underline{k}\cdot\underline{p})\frac{\underline{k}^2}{k_0^2} + 4p_\ell\frac{\underline{k}^2\underline{k}^2}{k_0^2}\right]$$

$$= -4p_\ell ig^2 i f^{abc}T^bT^c \int d^4k\ \frac{2}{3}\frac{\underline{k}^2}{k_0^2 k^4}$$

$$= -\frac{8}{3}p_\ell ig^2 i f^{abc}T^bT^c (2\pi)^4 I_8$$

$$= (2p_\ell)\, 4 i f^{abc}T^bT^c L \qquad \ldots(G81)$$

Since the axial gauge propagator has purely spatial components, we have immediately:

$$\Gamma^a_{0(iv)+(v)}(p)\Big|_\infty = 0 \qquad \ldots(G82)$$

and

$$\Gamma^a_{\ell(iv)+(v)}(p)\Big|_\infty = 2ig^2\left(d^{abc}T^bT^c + \frac{1}{n}T^a\right)\int \frac{d^4k}{k^2[(p+k)^2-m^2]}(2p+k)_m\left[\delta_{\ell m} - \frac{k_\ell k_m}{k_0^2}\right]\Big|_\infty$$

$$= 2ig^2\left(d^{abc}T^bT^c + \frac{1}{n}T^a\right)\int \frac{d^4k}{k^4}\left[1-\frac{2k\cdot p}{k^2}\cdots\right]\left[2p_\ell + k_\ell - k_\ell\frac{2\underline{p}\cdot\underline{k}+\underline{k}^2}{k}\right]\Big|_\infty$$

$$= 2ig^2\left(d^{abc}T^bT^c + \frac{1}{n}T^a\right)\int \frac{d^4k}{k^4}\left[2p_\ell + 2k_\ell\frac{k\cdot p}{k^2} - 2k_\ell\frac{\underline{k}\cdot\underline{p}}{k_0^2} - 2k_\ell\frac{\underline{k}^2\,\underline{k}\cdot\underline{p}}{k^2k_0^2}\right]$$

$$= 4p_\ell ig^2\left(d^{abc}T^bT^c + \frac{1}{n}T^a\right)(2\pi)^4\left[I_1 + I_2 - I_8 - I_9\right]$$

$$= (2p_\ell)(-2)\left(d^{abc}T^bT^c + \frac{1}{n}T^a\right) L \qquad \ldots(G83)$$

Adding (G77), (G80) and (G82) gives:

$$\Gamma^a_{0\,(total)}(p)\Big|_\infty = 2p_0[-6\,T^bT^aT^b + 6if^{abc}T^bT^c]\,L \qquad \text{...(G84a)}$$

$$= 2p_0\left[\frac{-3(n^2-1)}{n}\right]T^aL \qquad \text{...(G84b)}$$

Adding (G78), (G81) and (G83) gives

$$\Gamma^a_{\ell\,(total)}(p)\Big|_\infty = 2p_\ell[-2T^bT^aT^b + 4i\,f^{abc}T^bT^c - 2(d^{abc}T^bT^c + \tfrac{1}{n}T^a)]\,L \qquad \text{...(G85a)}$$

$$= 2p_\ell\left[\frac{-3(n^2-1)}{n}\right]T^aL \qquad \text{...(G85b)}$$

where (B16), (B17) and (B18) have been used.

So
$$\Gamma^a_{\lambda\,(total)}(p)\Big|_\infty = 2p_\lambda\left[\frac{-3(n^2-1)}{n}\right]T^aL \qquad \text{...(G86)}$$

which is covariant.

FERMION VERTEX CORRECTION:

$$\Gamma^a_{\lambda(i)}(p)\Big|_\infty = i\,g^2\,T^bT^aT^b\int\frac{d^4k}{k^2[(p+k)^2-m^2]^2}\,\gamma_i(\not p+\not k+m)\gamma_\lambda(\not p+\not k+m)\gamma_j\,\times$$
$$\times\left[\delta_{ij} - \frac{k_ik_j}{k_0^2}\right]\Big|_\infty \qquad \text{...(G87)}$$

leading to

$$\Gamma^a_{\lambda(i)}(p)\Big|_\infty = i\,g^2T^bT^aT^b\int\frac{d^4k}{k^6}\,\gamma_i\not k\,\gamma_\lambda\not k\,\gamma_j\left[\delta_{ij} - \frac{k_ik_j}{k_0^2}\right]$$

So
$$\Gamma^a_{o(i)}(p)\Big|_\infty = i\,g^2\,T^bT^aT^b\int\frac{d^4k}{k^6}\Big\{3\gamma_0(k_0^2+\underline{k}^2) - 2\gamma_0\,\underline{\gamma}.\underline{k}$$
$$-\frac{1}{k_0^2}\left[\gamma_0\,\underline{k}^2(k_0^2+\underline{k}^2) + 2k_0\,\underline{k}^2\underline{\gamma}.\underline{k}\right]\Big\}$$

$$= i\,g^2T^bT^aT^b\int\frac{d^4k}{k^6}\,\gamma_0\left[3k_0^2 + 2\underline{k}^2 - \frac{\underline{k}^4}{k_0^2}\right]$$

$$= \gamma_0\,i\,g^2T^bT^aT^b(2\pi)^{-4}[3I_3 + 2I_2 - I_9]$$

$$= \gamma_0\,3\,T^bT^aT^b\,L \qquad \text{...(G88)}$$

where (E32) and (E33) have been used.

and

$$\Gamma^a_{\ell(i)}(p)\Big|_\infty = i g^2 T^b T^a T^b \int \frac{d^4k}{k^6} \{ -k^2\gamma_\ell + 6\,\gamma_0 k_0 k_\ell - 2\,\underline{\gamma}.\underline{k}\,k_\ell$$

$$- \frac{1}{k_0^2}[2\,k_0^2\underline{\gamma}.\underline{k}\,k_\ell - k^2\underline{k}^2\gamma_\ell + 2\,\underline{k}^2\gamma_0 k_0 k_\ell]\}$$

$$= -i g^2 T^b T^a T^b \int \frac{d^4k}{k^6}[-k^2\gamma_\ell - 4\,\underline{\gamma}.\underline{k}\,k_\ell + \underline{k}^2\gamma_\ell - \frac{k^4}{k_0^2}\gamma_\ell]$$

$$= -\gamma_\ell i g^2 T^b T^a T^b (2\pi)^4 [I_1 + \tfrac{1}{3} I_2 + I_9]$$

$$= \gamma_\ell\, 3\, T^b T^a T^b\, L \qquad \ldots(G89)$$

where (E35) and (E36) have been used.

$$\Gamma^a_{\lambda(ii)}(p)\Big|_\infty = -i g^2 i f^{abc} T^b T^c \int \frac{d^4k}{(k^2-m^2)(p-k)^4}\,\gamma_m(\not{k}+m)\gamma_n \times$$

$$\times [(2p-2k)_\lambda \delta_{rs} - (p-k)_r\delta_{s\lambda} - (p-k)_s\delta_{\lambda r}] \times$$

$$\times [\delta_{ms} - \frac{(p-k)_m(p-k)_s}{(p-k)_0^2}][\delta_{rn} - \frac{(p-k)_r(p-k)_n}{(p-k)_0^2}]\Big|_\infty \qquad \ldots(G90)$$

leading to

$$\Gamma^a_{\lambda(ii)}(p)\Big|_\infty = -i g^2 i f^{abc} T^b T^c \int \frac{d^4k}{k^6}\,\gamma_m\not{k}\gamma_n[-2\,k_\lambda\delta_{rs} + k_r\delta_{s\lambda} + k_s\delta_{\lambda r}]\times$$

$$\times[\delta_{ms} - \frac{k_m k_s}{k_0^2}][\delta_{rn} - \frac{k_r k_n}{k_0^2}]$$

So

$$\Gamma^a_{0(ii)}(p)\Big|_\infty = -i g^2 i f^{abc} T^b T^c \int \frac{d^4k}{k^6}\,2k_0\,\gamma_m\not{k}\gamma_n[\delta_{ms} - \frac{k_m k_s}{k_0^2}][\delta_{ns} - \frac{k_n k_s}{k_0^2}]$$

$$= -i g^2 i f^{abc} T^b T^c \int \frac{d^4k}{k^6}\,2k_0\,[-\gamma_0 k_0\gamma_m\gamma_n + 2\,k_m\gamma_n + \underline{\gamma}.\underline{k}\,\gamma_m\gamma_n]\times$$

$$\times[\delta_{mn} - 2\frac{k_m k_n}{k_0^2} + \frac{\underline{k}^2 k_m k_n}{k_0^4}]$$

$$= i g^2 i f^{abc} T^b T^c\,\gamma_0 \int \frac{d^4k}{k^6}\,2k_0^2\,[\underline{\gamma}^2 - 2\frac{(\underline{\gamma}.\underline{k})^2}{k_0^2} + \frac{\underline{k}^2}{k_0^4}(\underline{\gamma}.\underline{k})^2]$$

$$= \gamma_0 2i g^2 i f^{abc} T^b T^c (2\pi)^4[-3\,I_3 + 2I_2 - I_9]$$

$$= \gamma_0\,(-3)\,i f^{abc} T^b T^c\, L \qquad \ldots(G91)$$

where (E22) has been used.

and

$$\Gamma^a_{\ell(ii)}(p) = -i g^2 i f^{abc} T^b T^c \int \frac{d^4k}{k^6}\,\gamma_m\not{k}\gamma_n[-2k_\ell\delta_{rs} + k_r\delta_{\ell s} + k_s\delta_{\ell r}]\times$$

$$\times[\delta_{ms} - \frac{k_m k_s}{k_0^2}][\delta_{rn} - \frac{k_r k_n}{k_0^2}]$$

$$= i g^2 i f^{abc} T^b T^c \int \frac{d^4k}{k^6} \left[-\gamma_0 k_0 \gamma_m \gamma_n + 2 k_m \gamma_n + \underline{\gamma} \cdot \underline{k}\, \gamma_m \gamma_n \right] \times$$

$$\times \left[2 k_\ell \delta_{mn} - k_n \delta_{m\ell} - k_m \delta_{\ell n} - \frac{2}{k_0^2} k_m k_n k_\ell + \frac{k^2}{k_0^2} (k_n \delta_{m\ell} + k_m \delta_{n\ell}) \right]$$

$$= i g^2 i f^{abc} T^b T^c \int \frac{d^4k}{k^6} \left[2 k_\ell \underline{\gamma} \cdot \underline{k} + 2 \underline{k}^2 \gamma_\ell + 2 \frac{k^2}{k_0^2} k\, \underline{\gamma} \cdot \underline{k} - 2 \frac{k^4}{k_0^2} \gamma_\ell \right]$$

$$= i g^2 i f^{abc} T^b T^c \gamma_\ell \int \frac{d^4k}{k^6} \left[\frac{8}{3} \underline{k}^2 - \frac{4}{3} \frac{k^4}{k_0^2} \right]$$

$$= \gamma_\ell \frac{4}{3} i g^2 i f^{abc} T^b T^c (2\pi)^{-4} \left[2 I_2 - I_9 \right]$$

$$= \gamma_\ell (-3) i f^{abc} T^b T^c L \qquad \ldots (G92)$$

where (E22), (E23) and (E24) have been used.

Adding (G88) and (G91) gives

$$\Gamma^a_{\substack{0 \\ (total)}}(p)\Big|_\infty = \gamma_0 \; 3 \left[T^b T^a T^b - i f^{abc} T^b T^c \right] L \qquad \ldots (G93a)$$

$$= \gamma_0 \left[\frac{-3(n^2-1)}{2n} \right] T^a L \qquad \ldots (G93b)$$

Adding (G89) and (G92) gives

$$\Gamma^a_{\substack{\ell \\ (total)}}(p)\Big|_\infty = \gamma_\ell \; 3 \left[T^b T^a T^b - i f^{abc} T^b T^c \right] L \qquad \ldots (G94a)$$

$$= \gamma_\ell \left[\frac{-3(n^2-1)}{2n} \right] T^a L \qquad \ldots (G94b)$$

Thus the total correction is covariant:

$$\Gamma^a_{\substack{\lambda \\ (total)}}(p)\Big|_\infty = \gamma_\lambda \; 3 \left[T^b T^a T^b - i f^{abc} T^b T^c \right] L \qquad \ldots (G95a)$$

$$= \gamma_\lambda \left[\frac{-3(n^2-1)}{2n} \right] T^a L \qquad \ldots (G95b)$$

where (B16) and (B17) have been used.

Note that, in the fermion case, not only is the covariance of the total correction more transparent than in the scalar meson case, but also each diagram gives a separately covariant contribution.

COULOMB GAUGE

In this gauge it is enlightening to analyse each calculation in terms of the longitudinal component of the propagator $\Delta_{00}(\mathbf{k})$ and the transverse component $\Delta_{ij}(\mathbf{k})$, represented by ||||||| and +++++ respectively. Terms indicated by $\big|_L$, $\big|_T$ etc., are analysed separately.

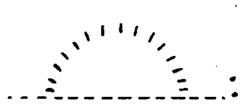
SCALAR MESON SELF-ENERGY:

$$\Pi(p)\big|_\infty = \Pi_{(i)}(p)\big|_\infty$$

$$= -i g^2 C_\phi \int \frac{d^\ell k}{k^2[(p+k)^2 - m^2]} \left[(2p+k)_0^2 \frac{k^2}{\mathbf{k}^2} + (2p+k)_i (2p+k)_j \left(\delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2} \right) \right]\bigg|_\infty$$

$$= -i g^2 C_\phi \int \frac{d^\ell k}{k^4} \left[1 - \frac{2k.p}{k^2} - \frac{p^2 - m^2}{k^2} + \frac{4(k.p)^2}{k^4} + \cdots \right] \times$$

$$\times \left[\frac{k^2}{\mathbf{k}^2} (k_0^2 + 4 k_0 p_0 + 4 p_0^2) \Big|_L + 4 \left(\mathbf{p}^2 - \frac{(\mathbf{k}.\mathbf{p})^2}{\mathbf{k}^2} \right) \Big|_T \right] \bigg|_\infty \qquad \ldots \text{(G96)}$$



L

$$: \; -i g^2 C_\phi \int \frac{d^\ell k}{k^4} \left[-(p^2 - m^2) \frac{k^2}{\mathbf{k}^2} + 4 p_0^2 \left(\frac{k_0^4}{\mathbf{k}^2 \mathbf{k}^2} - 2 \frac{k_0^2}{\mathbf{k}^2} + \frac{k^2}{\mathbf{k}^2} \right) + \frac{4}{3} \mathbf{p}^2 \frac{k_0^2}{\mathbf{k}^2} \right]$$

$$= -i g^2 C_\phi (2\pi)^{-4} \left[-(p^2 - m^2) I_{12} + 4 p_0^2 (I_{13} - 2 I_{12} + 4 I_{11}) + \frac{4}{3} \mathbf{p}^2 I_3 \right]$$

$$= \left[-2 p_0^2 - \frac{2}{3} \mathbf{p}^2 - m^2 \right] C_\phi L \qquad \ldots \text{(G97)}$$

T

$$: \; -i g^2 C_\phi \int \frac{d^\ell k}{k^4} \left[4 \mathbf{p}^2 - 4 \frac{(\mathbf{p}.\mathbf{k})^2}{\mathbf{k}^2} \right]$$

$$= -i g^2 C_\phi \int \frac{d^\ell k}{k^4} \frac{8}{3} \mathbf{p}^2$$

$$= -\frac{8}{3} \mathbf{p}^2 \, i g^2 C_\phi (2\pi)^{-4} I_1$$

$$= \frac{8}{3} \mathbf{p}^2 C_\phi L \qquad \ldots \text{(G98)}$$

Adding (G97) and (G98) gives

$$\Pi(p)\big|_\infty = -(2 p^2 + m^2) C_\phi L \qquad \ldots \text{(G99)}$$

FERMION SELF-ENERGY:

$$\Sigma(p)\big|_\infty = -i g^2 C_\psi \int \frac{d^4k}{k^2[(p+k)^2-m^2]} \times$$
$$\times \left[\gamma_0 (\not p + \not k - m)\gamma_0 \frac{k^2}{\mathbf{k}^2} + \gamma_i (\not p + \not k - m)\gamma_j \left(\delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2}\right)\right]\bigg|_\infty$$
$$= -i g^2 C_\psi \int \frac{d^4k}{k^4}\left[1 - \frac{2k\cdot p}{k^2} + \cdots\right]\times$$
$$\times \left\{ \frac{k^2}{\mathbf{k}^2}\left[\gamma_0(p_0+k_0) + \boldsymbol{\gamma}\cdot(\mathbf{p}+\mathbf{k}) - m\right]\Big|_L \right.$$
$$\left. + 2\left[\gamma_0(p_0+k_0) - \boldsymbol{\gamma}\cdot\mathbf{k} - \frac{(\mathbf{k}\cdot\mathbf{p})(\boldsymbol{\gamma}\cdot\mathbf{k})}{\mathbf{k}^2} + m\right]\Big|_T \right\}\bigg|_\infty \quad \ldots\text{(G100)}$$

where (E20), (E21), (E24) and (E25) have been used.

L

$$: -i g^2 C_\psi \int \frac{d^4k}{k^2 \mathbf{k}^2}\left[\gamma_0 p_0 - 2\gamma_0 p_0 \frac{k_0^2}{k^2} + \boldsymbol{\gamma}\cdot\mathbf{p} + 2\frac{\boldsymbol{\gamma}\cdot\mathbf{k}\,\mathbf{k}\cdot\mathbf{p}}{k^2} - m\right]$$
$$= -i g^2 C_\psi (2\pi)^{-4}\left[\gamma_0 p_0 (I_{11} - 2I_{12}) + \boldsymbol{\gamma}\cdot\mathbf{p}\left(I_{11} + \tfrac{2}{3}I_1\right) - m I_{11}\right]$$
$$= \left[-\tfrac{4}{3}\boldsymbol{\gamma}\cdot\mathbf{p} + 2m\right] C_\psi L \quad \ldots\text{(G101)}$$

T

$$: -i g^2 C_\psi \int \frac{d^4k}{k^4}\left[2\gamma_0 p_0 - 4\gamma_0 p_0 \frac{k_0^2}{k^2} - 4\frac{\boldsymbol{\gamma}\cdot\mathbf{k}\,\mathbf{k}\cdot\mathbf{p}}{k^2} + 2m - 2\frac{\boldsymbol{\gamma}\cdot\mathbf{k}\,\mathbf{k}\cdot\mathbf{p}}{\mathbf{k}^2}\right]$$
$$= -i g^2 C_\psi (2\pi)^{-4}\left[\gamma_0 p_0 (2I_1 - 4I_3) + \boldsymbol{\gamma}\cdot\mathbf{p}\left(-\tfrac{4}{3}I_2 - \tfrac{2}{3}I_1\right) + 2m I_1\right]$$
$$= \left[\gamma_0 p_0 + \tfrac{1}{3}\boldsymbol{\gamma}\cdot\mathbf{p} + 2m\right] C_\psi L \quad \ldots\text{(G102)}$$

Adding (G101) and (G102) gives

$$\Sigma(p)\big|_\infty = (\not p + 4m)\, C_\psi L \quad \ldots\text{(G103)}$$

VECTOR MESON SELF-ENERGY:

$$\Pi^{aa'}_{\kappa\kappa'}(p)\Big|_\infty = -i g^2 C_A \delta^{aa'} \int d^{2\ell}q\, d^{2\ell}r\, \delta(p+q+r) \times$$
$$\times \Big\{ \tfrac{1}{2} [(q-r)_\kappa \eta_{\lambda\mu} + (r-p)_\lambda \eta_{\mu\kappa} + (p-q)_\mu \eta_{\kappa\lambda}][(q-r)_{\kappa'} \eta_{\lambda'\mu'} + (r-p)_{\lambda'} \eta_{\mu'\kappa'} + (p-q)_{\mu'} \eta_{\kappa'\lambda'}] \times$$
$$\times \Delta^{\lambda\lambda'}(q)\, \Delta^{\mu\mu'}(r)$$
$$+ \frac{1}{q^2 r^2} (q_\kappa - q_0 \eta_{\kappa 0})(r_{\kappa'} - r_0 \eta_{\kappa' 0}) \Big\}\Big|_\infty \qquad \text{...(G104)}$$

First we show that $\Pi^{aa'}_{\kappa\kappa'}(p)\big|_\infty$ is doubly transverse:

$$p^\kappa p^{\kappa'} \Pi^{aa'}_{\kappa\kappa'}(p)\Big|_\infty = -\tfrac{1}{2} i g^2 C_A \delta^{aa'} \int d^{2\ell}q\, d^{2\ell}r\, \delta(p+q+r) \times$$
$$\times \Big\{ [(r^2-q^2)\eta_{\lambda\mu} + q_\lambda q_\mu - r_\lambda r_\mu][(r^2-q^2)\eta_{\lambda'\mu'} + q_{\lambda'} q_{\mu'} - r_{\lambda'} r_{\mu'}]\, \Delta^{\lambda\lambda'}(q)\, \Delta^{\mu\mu'}(r)$$
$$+ \frac{2}{q^2 r^2} (\underline{p}\cdot\underline{q})(\underline{p}\cdot\underline{r}) \Big\}\Big|_\infty \qquad \text{...(G105)}$$

where (F9) has been used.

$\Delta_{\lambda\lambda'}(q)\, \Delta_{\mu\mu'}(r)$ has as its only non-zero terms:

$\Delta_{00}(q)\, \Delta_{00}(r)$, $\Delta_{\ell\ell'}(q)\, \Delta_{00}(r)$, $\Delta_{00}(q)\, \Delta_{mm'}(r)$, $\Delta_{\ell\ell'}(q)\, \Delta_{mm'}(r)$.

Equation (G105) will be analysed term by term, remembering that:

$\Delta_{00}(q)$ is the longitudinal part of the propagator (L).

$\Delta_{\ell\ell'}(q)$ is the transverse part of the propagator (T).

$$LL \quad : \int d^{2\ell}q\, d^{2\ell}r\, \delta(p+q+r) \frac{[(r^2-q^2) + q_0^2 + r_0^2]^2}{\underline{q}^2 \underline{r}^2}\Bigg|_\infty$$
$$= 2 \int d^{2\ell}q\, d^{2\ell}r\, \delta(p+q+r) \left[\frac{\underline{r}^2}{\underline{q}^2} - 1 \right]\Big|_\infty$$
$$= 2 \int d^{2\ell}q \left[2 \frac{\underline{q}\cdot\underline{p}}{\underline{q}^2} + \frac{\underline{p}^2}{\underline{q}^2} \right]\Big|_\infty$$
$$= 0 \qquad \text{...(G106)}$$

$$LT+TL \;:\; 2\int d^{2\ell}q\, d^{2\ell}r\, \delta(p+q+r)\frac{1}{r^2 q^2}(q_\ell q_0 - r_\ell r_0)(q_{\ell'} q_0 - r_{\ell'} r_0)\left(\delta_{\ell\ell'} - \frac{q_\ell q_{\ell'}}{q^2}\right)\Big|_\infty$$

$$= 2\int d^{2\ell}q\, d^{2\ell}r\, \delta(p+q+r)\frac{r_0^2}{q^2}\left[1 - \frac{(\underline{q}\cdot\underline{r})^2}{\underline{q}^2\underline{r}^2}\right]\Big|_\infty$$

$$= -2\int d^{2\ell}q\, d^{2\ell}r\, \delta(p+q+r)\frac{r_0^2}{q^2}\frac{(\underline{q}\cdot\underline{r})^2}{\underline{q}^2\underline{r}^2}\Big|_\infty \qquad \text{...(G107)}$$

$$TT \;:\; \int dq\, dr\, \delta(p+q+r)\frac{1}{q^2 r^2}\left[\delta_{\ell\ell'} - \frac{q_\ell q_{\ell'}}{\underline{q}^2}\right]\left[\delta_{mm'} - \frac{r_m r_{m'}}{\underline{r}^2}\right]\times$$

$$\times\left[-(r^2-q^2)\delta_{\ell m} + q_\ell q_m - r_\ell r_m\right]\left[-(r^2-q^2)\delta_{\ell' m'} + q_{\ell'} q_{m'} - r_{\ell'} r_{m'}\right]\Big|_\infty$$

$$= \int d^{2\ell}q\, d^{2\ell}r\, \delta(p+q+r)\frac{1}{q^2 r^2}\times$$

$$\times\Big\{ 3(r^2-q^2) + 2(r^2-q^2)(\underline{r}^2-\underline{q}^2) + \underline{q}^4 + \underline{r}^4 - 2(\underline{q}\cdot\underline{r})^2$$

$$-\frac{1}{\underline{q}^2}\left[(r^2-q_0^2)q_m + (\underline{q}\cdot\underline{r})r_m\right]\left[(r^2-q_0^2)q_m + (\underline{q}\cdot\underline{r})r_m\right]$$

$$-\frac{1}{\underline{r}^2}\left[(r_0^2-q^2)r_\ell - (\underline{q}\cdot\underline{r})q_\ell\right]\left[(r_0^2-q^2)r_\ell - (\underline{q}\cdot\underline{r})q_\ell\right]$$

$$+\frac{1}{\underline{q}^2\underline{r}^2}(\underline{q}\cdot\underline{r})^2(r_0^2-q_0^2)^2 \Big\}\Big|_\infty$$

$$= 2\int d^{2\ell}q\, d^{2\ell}r\, \delta(p+q+r)\times$$

$$\times\Big\{\underline{q}^4 - (\underline{q}\cdot\underline{r})^2 - \frac{1}{\underline{q}^2}\left[(r^2-q_0^2)^2\underline{q}^2 + 2(\underline{q}\cdot\underline{r})^2(r^2-q_0^2) + (\underline{q}\cdot\underline{r})^2\underline{r}^2\right.$$

$$+\frac{(\underline{q}\cdot\underline{r})^2}{\underline{q}^2\underline{r}^2}\left[r_0^4 - q_0^2 r_0^2\right]\Big\}\Big|_\infty \qquad \text{...(G108)}$$

where (G65) and (G66) have been used.

Adding (G106), (G107), (G108) and the ghost term gives

$$p^\kappa p^{\kappa'}\Pi^{aa'}_{\kappa\kappa'}(p)\Big|_\infty = -ig^2 C_A\,\delta^{aa'}\int d^{2\ell}q\, d^{2\ell}r\, \delta(p+q+r)\frac{1}{r^2}\left[q^2 + 2\underline{q}^2\right]\Big|_\infty$$

where extensive use of symmetry has been made.

Thus

$$p^\kappa p^{\kappa'}\Pi^{aa'}_{\kappa\kappa'}(p)\Big|_\infty = -ig^2 C_A\,\delta^{aa'}\int\frac{d^{2\ell}q}{q^2}\left[q^2 + 2\underline{q}^2\right]\left[1 - \frac{2q\cdot p}{q^2} - \frac{p^2}{q^2} + \frac{4(q\cdot p)^2}{q^4}\right.$$

$$\left. + 4p^2\frac{q\cdot p}{q^4} - 8\frac{(q\cdot p)^3}{q^6} + \frac{p^4}{q^4} - 12p^2\frac{(q\cdot p)^2}{q^6} + 16\frac{(q\cdot p)^4}{q^8} + \dots\right]\Big|_\infty$$

$$= -ig^2 C_A \delta^{aa'} \int d^{2\omega}q \left[-p^4 \frac{1}{q^4} + 12 p^2 p_0^2 \frac{q_0^2}{q^6} + 12 p^2 \frac{(\mathbf{q}\cdot\mathbf{p})^2}{q^6} - 16 p_0^4 \frac{q_0^4}{q^8} \right.$$
$$- 96 p_0^2 \frac{q_0^2(\mathbf{q}\cdot\mathbf{p})^2}{q^8} - 16 \frac{(\mathbf{q}\cdot\mathbf{p})^4}{q^8} - 2 p^4 \frac{\mathbf{q}^2}{q^6} + 24 p^2 p_0^2 \frac{q_0^2 \mathbf{q}^2}{q^8} + 24 p^2 \frac{(\mathbf{q}\cdot\mathbf{p})^2 \mathbf{q}^2}{q^8}$$
$$\left. - 32 p_0^4 \frac{q_0^4 \mathbf{q}^2}{q^{10}} - 192 p_0^2 \frac{q_0^2(\mathbf{q}\cdot\mathbf{p})^2 \mathbf{q}^2}{q^{10}} - 32 \frac{(\mathbf{q}\cdot\mathbf{p})^4 \mathbf{q}^2}{q^{10}} \right]$$
$$= ig^2 C_A \delta^{aa'} (2\pi)^{-4} \{ p_0^4 [-I_1 + 12 I_3 - 16 I_6 - 2 I_2 + 24 I_5 - 32 I_{18}]$$
$$+ p_0^2 \mathbf{p}^2 [2I_1 - 12 I_3 + 8 I_2 - 56 I_5 + 8 I_4 - 64 I_{17}]$$
$$+ \mathbf{p}^4 [-I_1 - 6 I_2 - \tfrac{56}{5} I_4 - \tfrac{32}{5} I_{16}] \}$$
$$= 0 \qquad \ldots (G109)$$

So $\Pi^{aa'}_{KK'}(p)\big|_\infty$ is doubly transverse. As discussed in the text it takes the form (suppressing group indices):

$$\Pi_{KK'}(p)\big|_\infty = (-\eta_{KK'} p^2 + p_K p_{K'})\, \Pi_C + (p^2 \eta_{K0} - p_0 p_K)(p^2 \eta_{K'0} - p_0 p_{K'})\, \Pi_N$$
$$+ [(p^2 \eta_{K0} - p_0 p_K)\eta_{K'} + (p^2 \eta_{K'0} - p_0 p_{K'}) \eta_K]\, \Pi_M \qquad \ldots (G110)$$

By power counting Π_N does not contribute to the divergent part, and so only two simultaneous equations are needed to find the scalars Π_C and Π_M.

We choose to evaluate

$$\Pi_{00}(\mathbf{p}^2)\big|_\infty = \mathbf{p}^2 (\Pi_C - 2\, \Pi_M) \qquad \ldots (G111)$$

$$\Pi_{kk}(p_0^2, \mathbf{p}^2)\big|_\infty = (3 p_0^2 - 2 \mathbf{p}^2)\, \Pi_C \qquad \ldots (G112)$$

There is also

$$\Pi_K{}^K(p, \mathbf{p})\big|_\infty = -3 p^2 \Pi_C - 2 \mathbf{p}^2 \Pi_M \qquad \ldots (G113)$$

The factors $(\mathbf{p}^2)$ and $(3p_0^2 - 2\mathbf{p}^2)$ serve as a check on calculations.

There is no ghost field term in $\Pi_{oo}(p)\big|_\infty$ since $\Lambda_o^{abc}(p) = 0$.

Thus

$$\Pi_{oo}(p)\Big|_\infty = -\frac{1}{2} i g^2 C_A \int d^{4}q\, d^{4}r\; \delta(p+q+r)\Lambda_{o\lambda u}(p,q,r)\Lambda_{o\lambda' u'}(p,q,r)\Delta^{\lambda\lambda'}(q)\Delta^{uu'}(r)\Big|_\infty$$

$$= -\frac{1}{2} i g^2 C_A \int d^{4}q\, d^{4}r\; \delta(p+q+r) \times$$

$$\times \Big\{ 2\Lambda_{o\ell o}(p,q,r)\,\Delta_{\ell\ell'}(q)\,\Delta_{oo}(r)\,\Lambda_{o\ell' o}(p,q,r)\Big|_{LT}$$

$$+\Lambda_{o\ell m}(p,q,r)\,\Delta_{\ell\ell'}(q)\Delta_{mm'}(r)\Lambda_{o\ell' m'}(p,q,r)\Big|_{TT}\Big\}\Big|_\infty \qquad \ldots(G114)$$

LT

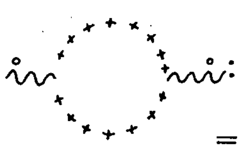
$$: \; -i g^2 C_A \int d^{4}q\, d^{4}r\; \delta(p+q+r)\frac{1}{q^2\mathbf{r}^2}(r-p)_\ell (r-p)_{\ell'}\Big(\delta_{\ell\ell'} - \frac{q_\ell q_{\ell'}}{\mathbf{q}^2}\Big)\Big|_\infty$$

$$= -i g^2 C_A \int d^{4}q\, d^{4}r\; \delta(p+q+r)\frac{1}{q^2\mathbf{r}^2}\Big[(\mathbf{r}-\mathbf{p})^2 - \frac{(\mathbf{q}\cdot(\mathbf{r}-\mathbf{p}))^2}{\mathbf{q}^2}\Big]\Big|_\infty$$

$$= -4 i g^2 C_A \int \frac{d^{4}q}{q^2\mathbf{q}^2}\Big[1 - \frac{2\mathbf{q}\cdot\mathbf{p}}{\mathbf{q}^2} + \cdots\Big]\Big[\mathbf{p}^2 - \frac{(\mathbf{q}\cdot\mathbf{p})^2}{\mathbf{q}^2}\Big]\Big|_\infty$$

$$= -\frac{8}{3}\mathbf{p}^2 i g^2 C_A (2\pi)^{-4} I_{11}$$

$$= \mathbf{p}^2\Big(-\frac{16}{3}\Big) C_A L \qquad \ldots(G115)$$



TT

$$: \; -\frac{1}{2} i g^2 C_A \int d^{4}q\, d^{4}r\; \delta(p+q+r)\frac{(q-r)_0^2}{q^2 r^2}\delta_{\ell m}\delta_{\ell' m'}\Big(\delta_{\ell\ell'} - \frac{q_\ell q_{\ell'}}{\mathbf{q}^2}\Big)\Big(\delta_{mm'} - \frac{r_m r_{m'}}{\mathbf{r}^2}\Big)\Big|_\infty$$

$$= -\frac{1}{2} i g^2 C_A \int d^{4}q\, d^{4}r\; \delta(p+q+r)\frac{(q-r)_0^2}{q^2 r^2}\Big[1 + \frac{(\mathbf{q}\cdot\mathbf{r})^2}{\mathbf{q}^2\mathbf{r}^2}\Big]\Big|_\infty$$

$$= -i g^2 C_A \int d^{4}q\, d^{4}r\; \delta(p+q+r) \times$$

$$\times\Big[\frac{q_0^2}{q^2 r^2} + \frac{(\mathbf{q}\cdot\mathbf{r})^2}{q^2\mathbf{q}^2\mathbf{r}^2} + \frac{(\mathbf{q}\cdot\mathbf{r})^2}{q^2\mathbf{q}^2 r^2} - \frac{q_0 r_0}{q^2 r^2} - \frac{q_0 r_0(\mathbf{q}\cdot\mathbf{r})^2}{q^2 r^2\mathbf{q}^2\mathbf{r}^2}\Big]\Big|_\infty$$

$$= -i g^2 C_A \int d^{4}q \times$$

$$\times\Big\{\frac{1}{q^4}\Big[1 - \frac{2q\cdot p}{q^2} - \frac{p^2}{q^2} + \frac{4(q\cdot p)^2}{q^4} + \cdots\Big]\Big[2\mathbf{q}^2 + 2(\mathbf{q}\cdot\mathbf{p}) + \frac{(\mathbf{q}\cdot\mathbf{p})^2}{\mathbf{q}^2} + q_0^2 + q_0 p_0\Big]$$

$$+\frac{1}{q^2\mathbf{q}^2}\Big[1 - \frac{2\mathbf{q}\cdot\mathbf{p}}{\mathbf{q}} - \frac{\mathbf{p}^2}{\mathbf{q}^2} + \frac{4(\mathbf{q}\cdot\mathbf{p})^2}{\mathbf{q}^4} + \cdots\Big]\Big[\mathbf{q}^2 + 2(\mathbf{q}\cdot\mathbf{p}) + \frac{(\mathbf{q}\cdot\mathbf{p})^2}{\mathbf{q}^2}\Big]$$

$$+\frac{1}{q^4\mathbf{q}^4}\Big[1 - \frac{2q\cdot p}{q^2} - \frac{2\mathbf{q}\cdot\mathbf{p}}{\mathbf{q}^2} - \frac{p^2}{q^2} - \frac{\mathbf{p}^2}{\mathbf{q}^2} + \frac{4(q\cdot p)^2}{q^4} + \frac{4(q\cdot p)(\mathbf{q}\cdot\mathbf{p})}{q^2\mathbf{q}^2} + \frac{4(\mathbf{q}\cdot\mathbf{p})^2}{\mathbf{q}^4}\cdots\Big]\times$$

$$\times\Big[q_0^2\mathbf{q}^4 + 2q_0^2\mathbf{q}^2(\mathbf{q}\cdot\mathbf{p}) + q_0^2(\mathbf{q}\cdot\mathbf{p})^2$$

$$+ q_0 p_0\mathbf{q}^4 + 2q_0 p_0\mathbf{q}^2(\mathbf{q}\cdot\mathbf{p}) + q_0 p_0(\mathbf{q}\cdot\mathbf{p})^2\Big]\Big\}\Big|_\infty$$

$$= -i g^2 C_A \int d^4 q \; \times$$

$$\times \Big\{ \frac{1}{q^4}\Big[-2 p^2 \frac{\mathbf{q}^2}{q^2} + 8 p_0^2 \frac{q_0^2 \mathbf{q}^2}{q^4} + 8 \frac{\mathbf{q}^2(\mathbf{q}\cdot\mathbf{p})^2}{q^4} + 4 \frac{(\mathbf{q}\cdot\mathbf{p})^2}{q^2} + \frac{(\mathbf{q}\cdot\mathbf{p})^2}{\mathbf{q}^2}$$

$$- (p^2 + 2p_0^2) \frac{q_0^2}{q^2} + 4 p_0^2 \frac{q_0^4}{q^4} + 4 \frac{q_0^2(\mathbf{q}\cdot\mathbf{p})^2}{q^4} \Big]$$

$$+ \frac{1}{q^2 \mathbf{q}^2}\Big[-\mathbf{p}^2 + \frac{(\mathbf{q}\cdot\mathbf{p})^2}{\mathbf{q}^2} \Big]$$

$$+ \frac{1}{q^4 \mathbf{q}^4}\Big[-(p^2 + 2p_0^2)\frac{q_0^2 \mathbf{q}^4}{q^2} - \mathbf{p}^2 q_0^2 \mathbf{q}^2 + 4 p_0^2 \frac{q_0^4 \mathbf{q}^4}{q^4}$$

$$+ 4 \frac{q_0^2 \mathbf{q}^4 (\mathbf{q}\cdot\mathbf{p})^2}{q^4} + q_0^2 (\mathbf{q}\cdot\mathbf{p})^2 \Big] \Big\}$$

$$= -i g^2 C_A (2\pi)^{-4} \Big[p_0^2 (-2 I_2 + 8 I_5 - 6 I_3 + 8 I_6)$$

$$+ \mathbf{p}^2 \big(\tfrac{8}{3} I_4 + \tfrac{10}{3} I_2 + \tfrac{1}{3} I_1 + 2 I_3 + \tfrac{8}{3} I_5 - \tfrac{2}{3} I_{11} - \tfrac{2}{3} I_{12} \big) \Big]$$

$$= \mathbf{p}^2 \; \tfrac{5}{3} \; C_A L \qquad \text{...(G116)}$$

Adding (G115) and (G116) gives

$$\Pi_{00}(p)\Big|_\infty = \mathbf{p}^2 \left(-\tfrac{11}{3}\right) C_A L \qquad \text{...(G117)}$$

There <u>is</u> a ghost field contribution to $\Pi_{kk}(p)\big|_\infty$.

Thus

$$\Pi_{kk}(p)\Big|_\infty = -\tfrac{1}{2} i g^2 C_A \int d^4 q \, d^4 r \; \delta(p+q+r) \times$$

$$\times \Big\{ \Lambda_{k\lambda\mu}(p,q,r)\, \Delta^{\lambda\lambda'}(q)\, \Delta^{\mu\mu'}(r)\, \Lambda_{k\lambda'\mu'}(p,q,r)$$

$$-2\, \Lambda_k(p,q,r)\, \Delta_\chi(q)\, \Delta_\chi(r)\, \Lambda_k(p,q,r) \Big\}\Big|_\infty$$

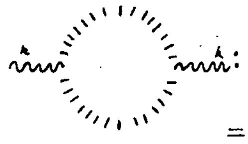
$$= -\tfrac{1}{2} i g^2 C_A \int d^4 q \, d^4 r \; \delta(p+q+r) \times$$

$$\times \Big\{ \Lambda_{k00}(p,q,r)\, \Delta_{00}(q)\, \Delta_{00}(r) \Lambda_{k00}(p,q,r)\Big|_{LL}$$

$$+ 2 \Lambda_{k0m}(p,q,r)\, \Delta_{00}(q)\, \Delta_{mm'}(r) \Lambda_{k0m'}(p,q,r)\Big|_{LT}$$

$$+ \Lambda_{klm}(p,q,r)\, \Delta_{ll'}(q)\, \Delta_{mm'}(r) \Lambda_{kl'm'}(p,q,r)\Big|_{TT}$$

$$-2\, \Lambda_k(p,q,r)\, \Delta_\chi(q) \Delta_\chi(r)\, \Lambda_k(p,q,r)\Big|_{GG} \Big\}\Big|_\infty \qquad \text{...(G118)}$$



L L

$$: -\tfrac{1}{2} i g^2 C_A \int d^4q\, d^4r\, \delta(p+q+r) \frac{(\underset{\sim}{q}-\underset{\sim}{r})^2}{\underset{\sim}{q}^2 \underset{\sim}{r}^2}\Bigg|_\infty$$

$$= -i g^2 C_A \int d^4q\, d^4r\, \delta(p+q+r) \frac{\underset{\sim}{q}^2 - \underset{\sim}{q}\cdot\underset{\sim}{r}}{\underset{\sim}{q}^2 \underset{\sim}{r}^2}\Bigg|_\infty$$

$$= i g^2 C_A \int d^4q\, d^4r\, \delta(p+q+r) \frac{\underset{\sim}{q}\cdot\underset{\sim}{r}}{\underset{\sim}{q}^2 \underset{\sim}{r}^2}\Bigg|_\infty \qquad \text{...(G119)}$$

Equation (G119) cancels the ghost term, and gives zero divergent contribution in any case.

L T

$$: -i g^2 C_A \int d^4q\, d^4r\, \delta(p+q+r) \frac{(r-p)_0^2}{q^2 r^2} \delta_{mm'} \left(\delta_{mm'} - \frac{r_m r_{m'}}{\underset{\sim}{r}^2}\right)\Bigg|_\infty$$

$$= -2i g^2 C_A \int d^4q\, d^4r\, \delta(p+q+r) \frac{r_0^2 - 2 r_0 p_0 + p_0^2}{\underset{\sim}{q}^2 r^2}\Bigg|_\infty$$

$$= -2i g^2 C_A \int \frac{d^4q}{\underset{\sim}{q}^2 q^2} \left[1 - \frac{2q\cdot p}{q^2} - \frac{p^2}{q^2} + \frac{4(q\cdot p)^2}{q^4} + \cdots\right]\left[q_0^2 + 4 q_0 p_0 + 4 p_0^2\right]\Bigg|_\infty$$

$$= -2 i g^2 C_A \int d^4q \left[(-9 p_0^2 + p^2)\frac{q_0^2}{\underset{\sim}{q}^2 q^4} + 4 p \frac{q_0^4}{\underset{\sim}{q}^2 q^6} + 4 \frac{q_0^2 (q\cdot p)^2}{\underset{\sim}{q}^2 q^6} + 4 p_0^2 \frac{1}{\underset{\sim}{q}^2 q^2}\right]$$

$$= -2 i g^2 C_A (2\pi)^{-4}\left[p_0^2(-9 I_{12} + 4 I_{13} + 4 I_{11}) + p^2\left(I_{12} + \tfrac{4}{3} I_3\right)\right]$$

$$= -\left(4 p_0^2 + \tfrac{4}{3} p^2\right) C_A L \qquad \text{...(G120)}$$

T T

$$: -\tfrac{1}{2} i g^2 C_A \int d^4q\, d^4r\, \delta(p+q+r) \frac{1}{q^2 r^2} \left(\delta_{\ell\ell'} - \frac{q_\ell q_{\ell'}}{\underset{\sim}{q}^2}\right)\left(\delta_{mm'} - \frac{r_m r_{m'}}{\underset{\sim}{r}^2}\right) \times$$

$$\times \left[(q-r)_k \delta_{\ell m} + (r-p)_\ell \delta_{mk} + (p-q)_m \delta_{k\ell}\right]\left[(q-r)_k \delta_{\ell' m'} + (r-p)_{\ell'} \delta_{m'k} + (p-q)_{m'} \delta_{k\ell'}\right]\Bigg|_\infty$$

$$= -i g^2 C_A \int d^4q\, d^4r\, \delta(p+q+r) \frac{1}{q^2 r^2} \times$$

$$\times \Big\{ 2\left[2\underset{\sim}{q}^2 - (\underset{\sim}{q}\cdot\underset{\sim}{r}) - 2(\underset{\sim}{q}\cdot\underset{\sim}{p}) + \underset{\sim}{p}^2\right]$$

$$- \frac{1}{\underset{\sim}{q}^2}\left[\underset{\sim}{q}^2 \underset{\sim}{r}^2 + (\underset{\sim}{q}\cdot\underset{\sim}{r})^2 - 4(\underset{\sim}{q}\cdot\underset{\sim}{r})(\underset{\sim}{q}\cdot\underset{\sim}{p}) + (\underset{\sim}{q}\cdot\underset{\sim}{p})^2 + \underset{\sim}{p}^2 \underset{\sim}{q}^2\right]$$

$$+ \frac{1}{\underset{\sim}{q}^2 \underset{\sim}{r}^2}\left[-(\underset{\sim}{q}\cdot\underset{\sim}{r})(\underset{\sim}{q}\cdot\underset{\sim}{p})(\underset{\sim}{p}\cdot\underset{\sim}{r}) + \underset{\sim}{r}^2(\underset{\sim}{q}\cdot\underset{\sim}{p})^2\right] \Big\}\Bigg|_\infty^{(1)}$$

$$= -i g^2 C_A \int d^4q\, d^4r\, \delta(p+q+r)$$

$$\times \left[4\underset{\sim}{q}^2 - 4(\underset{\sim}{q}\cdot\underset{\sim}{p}) - 2(\underset{\sim}{q}\cdot\underset{\sim}{r}) + \underset{\sim}{p}^2 - \underset{\sim}{r}^2 - \frac{(\underset{\sim}{q}\cdot\underset{\sim}{r})^2}{\underset{\sim}{q}^2} + 4\frac{(\underset{\sim}{q}\cdot\underset{\sim}{r})(\underset{\sim}{q}\cdot\underset{\sim}{p})}{\underset{\sim}{q}^2} - \frac{(\underset{\sim}{q}\cdot\underset{\sim}{r})(\underset{\sim}{q}\cdot\underset{\sim}{p})(\underset{\sim}{p}\cdot\underset{\sim}{r})}{\underset{\sim}{q}^2 \underset{\sim}{r}^2}\right]\Bigg|_\infty$$

$$= -i g^2 C_A \int d^4q \Big\{ \frac{\underset{\sim}{p}^2}{q^4} - \frac{(\underset{\sim}{q}\cdot\underset{\sim}{p})^2}{q^4 \underset{\sim}{q}^2}$$

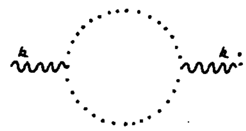
$$+ \frac{1}{q^4}\left[1 - \frac{2q\cdot p}{q^2} - \frac{p^2}{q^2} + \frac{4(q\cdot p)^2}{q^4} + \cdots\right] \times$$

$$\times \left[4\underset{\sim}{q}^2 - 10\,\underset{\sim}{q}\cdot\underset{\sim}{p} - \underset{\sim}{p}^2 - 5\frac{(\underset{\sim}{q}\cdot\underset{\sim}{p})^2}{\underset{\sim}{q}^2}\right]\Big\}\Bigg|_\infty$$

$$= -i\, g^2 C_A \int d^4q \left[-4\mathbf{p}^2 \frac{\mathbf{q}^2}{q^6} + 16 p_0^2 \frac{q_0^2 \mathbf{q}^2}{q^8} - 20 \frac{(\mathbf{q}\cdot\mathbf{p})^2}{q^6} - 6 \frac{(\mathbf{q}\cdot\mathbf{p})^2}{q^4 \mathbf{q}^2} + 16 \frac{\mathbf{q}^2(\mathbf{q}\cdot\mathbf{p})^2}{q^8} \right]$$

$$= -i\, g^2 C_A\, (2\pi)^{-4} \left[p_0^2(-4I_2 + 16I_5) + \mathbf{p}^2\left(-\tfrac{8}{3} I_2 - 2I_1 + \tfrac{16}{3} I_4\right) \right]$$

$$= \left(p_0^2 + \tfrac{10}{3}\mathbf{p}^2\right) C_A L \qquad \ldots(\text{G121})$$



GG

$$-i\, g^2 C_A \int d^4q\, d^4r\; \delta(p+q+r) \frac{\mathbf{q}\cdot\mathbf{r}}{q^2 r^2}\Big|_\infty \qquad \ldots(\text{G122})$$

Adding (G119), (G120), (G121) and (G122) gives

$$\Pi_{kk}(p)\big|_\infty = \left(-3p_0^2 + 2\mathbf{p}^2\right) C_A L \qquad \ldots(\text{G123})$$

From (G117) and (G123) we deduce both that

$$\Pi_k{}^k(p)\big|_\infty = \Pi_{00}(p)\big|_\infty - \Pi_{kk}(p)\big|_\infty$$

$$= \left(3p_0^2 - \tfrac{17}{3}\mathbf{p}^2\right) C_A L \qquad \ldots(\text{G124})$$

and, using (G111) and (G112), that

$$\Pi_C = \frac{1}{(3p_0^2 - 2\mathbf{p}^2)} \Pi_{kk}(p)\big|_\infty$$

$$= -C_A L \qquad \ldots(\text{G125})$$

and

$$\Pi_M = \tfrac{1}{2}\left[\frac{1}{(3p_0^2 - 2\mathbf{p}^2)} \Pi_{kk}(p)\big|_\infty - \frac{1}{\mathbf{p}^2} \Pi_{00}(p)\big|_\infty \right]$$

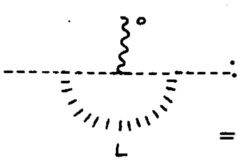
$$= \tfrac{4}{3} C_A L \qquad \ldots(\text{G126})$$

SCALAR VERTEX CORRECTION:

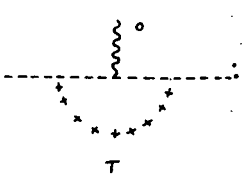
When working at zero momentum transfer the results are proportional to $2p_0$ and $2p_\ell$ respectively.

$$\Gamma^a_{\lambda(i)}(p)\Big|_{\substack{\infty \\ (p=p')}} = i g^2 T^b T^a T^b \int \frac{d^4k}{k^2[(p+k)^2-m^2]^2}(2p+2k)_\lambda \times$$
$$\times\left[(2p+k)_0^2\frac{k^2}{\underline{k}^2}+(2p+k)_i(2p+k)_j\left(\delta_{ij}-\frac{k_ik_j}{\underline{k}^2}\right)\right]\Big|_\infty \qquad \text{...(G127)}$$

$$\Gamma^a_{0(i)}(p)\Big|_\infty = i g^2 T^b T^a T^b \int \frac{d^4k}{k^6}\left[1-\frac{4k.p}{k^2}+\cdots\right](2p_0+2k_0)\times$$
$$\times\left[(k_0^2+4k_0p_0+4p_0^2)\frac{k^2}{\underline{k}^2}\Big|_L+\left(4\underline{p}^2-4\frac{(\underline{k}.\underline{p})^2}{\underline{k}^2}\right)\Big|_T\right]\Big|_\infty \qquad \text{...(G128)}$$



$$: \; i g^2 T^b T^a T^b \int d^4k\left[10p_0\frac{k_0^2}{k^4\underline{k}^2}-8p_0\frac{k_0^4}{k^6\underline{k}^2}\right]$$
$$= 2p_0\, i g^2 T^b T^a T^b (2\pi)^{-4}[5I_{12}-4I_{13}]$$
$$= (2p_0)\; 2\,T^b T^a T^b\, L \qquad \text{...(G129)}$$

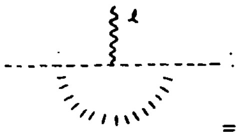


$$: \quad 0 \qquad \text{...(G130)}$$

Adding (G129) and (G130) gives

$$\Gamma^a_{0(i)}(p)\Big|_\infty = (2p_0)\; 2\,T^b T^a T^b\, L \qquad \text{...(G131)}$$

$$\Gamma^a_{\ell(i)}(p)\Big|_\infty = i g^2 T^b T^a T^b \int \frac{d^4k}{k^6}\left[1-\frac{4k.p}{k^2}+\cdots\right](2p_\ell+2k_\ell)\times$$
$$\times\left[(k_0^2+4k_0p_0+4p_0^2)\frac{k^2}{\underline{k}^2}\Big|_L+\left(4\underline{p}^2-4\frac{(\underline{k}.\underline{p})^2}{\underline{k}^2}\right)\Big|_T\right]\Big|_\infty \qquad \text{...(G132)}$$

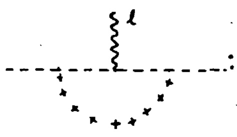


L

$$i g^2 T^b T^a T^b \int d^\ell k \left[2 p_\ell \frac{k_0^2}{k^4 \underline{k}^2} + 8 k_\ell \frac{k_0^2 (\underline{k} \cdot \underline{p})}{k^6 \underline{k}^2} \right]$$

$$= (2 p_\ell)\, i g^2 T^b T^a T^b (2\pi)^{-4} \left[I_2 + \tfrac{4}{3} I_3 \right]$$

$$= (2 p_\ell)\, \tfrac{2}{3}\, T^b T^a T^b\, L \qquad \ldots (G133)$$



T

$$0 \qquad \ldots (G134)$$

Adding (G133) and (G134) gives

$$\Gamma^a_{\ell(i)}(p)\Big|_\infty = (2 p_\ell)\, \tfrac{2}{3}\, T^b T^a T^b\, L \qquad \ldots (G135)$$

$$\Gamma^a_{\lambda(ii)}(p)\Big|_\infty = -i g^2 i f^{abc} T^b T^c \int \frac{d^\ell k}{(k^2 - m^2)} (p+k)_\mu (p+k)_\nu \Delta^{\mu\sigma}(p-k)\, \Delta^{\rho\nu}(p-k) \times$$
$(p = p')$

$$\times \left[-(2p - 2k)_\lambda \eta_{\rho\sigma} + (p-k)_\rho \eta_{\sigma\lambda} + (p-k)_\sigma \eta_{\lambda\rho} \right]\Big|_\infty$$

$$= -i g i f^{abc} T^b T^c \int \frac{d^\ell k}{[(p-k)^2 - m^2]} \times$$

$$\times \left\{ (2p - k)_0^2 \frac{1}{\underline{k}^4} \left[-2 k_\lambda + 2 k_0 \eta_{0\lambda} \right] \Big|_{LL} \right.$$

$$+ 2 (2p - k)_0 (2p - k)_n \frac{1}{k^2 \underline{k}^2} \left(\delta_{rn} - \frac{k_r k_n}{\underline{k}^2} \right) \left[k_r \eta_{0\lambda} + k_0 \eta_{\lambda r} \right] \Big|_{LT}$$

$$\left. + (2p - k)_m (2p - k)_n \frac{1}{k^4} \left(\delta_{ms} - \frac{k_m k_s}{\underline{k}^2} \right) \left(\delta_{rn} - \frac{k_r k_n}{\underline{k}^2} \right) \left[2 k_\lambda \delta_{rs} + k_r \eta_{s\lambda} k_s \eta_{\lambda r} \right] \Big|_{TT} \right\} \Big|_\infty$$

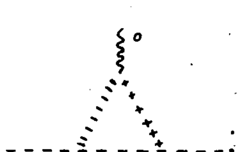
$$\ldots (G136)$$

$$\Gamma^a_{0(i)}(p)\Big|_\infty :$$

LL

$$-i g^2 i f^{abc} T^b T^c \int \frac{d^\ell k}{[(p-k)^2 - m^2]} (2p - k)_0^2 \frac{1}{\underline{k}^4} \left[-2 k_0 + 2 k_0 \right] \Big|_\infty$$

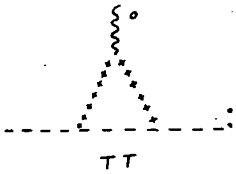
$$\equiv 0 \qquad \ldots (G137)$$



LT

$$-2 i g^2 i f^{abc} T^b T^c \int \frac{d^\ell k}{[(p-k)^2 - m^2]} (2p - k)_0 (2p - k)_n \frac{1}{k^2 \underline{k}^2} \left(\delta_{rn} - \frac{k_r k_n}{\underline{k}^2} \right) k_r \Big|_\infty$$

$$\equiv 0 \qquad \ldots (G138)$$



$$: \quad -ig^2 i f^{abc} T^b T^c \int \frac{d^4k}{\underline{k}^4[(p-k)^2-m^2]} 2k_0 \left[(2p-k)_s - \frac{\underline{k}\cdot(2\underline{p}-\underline{k})k_s}{\underline{k}^2} \right]^2 \Big|_\infty$$

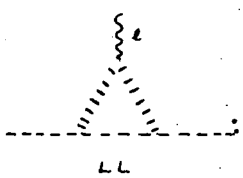
$$= -2ig^2 i f^{abc} T^b T^c \int \frac{d^4k}{k^6} k_0 \left[1 + \frac{2k\cdot p}{k^2} + \cdots \right] \left[4\underline{p}^2 - 4\frac{(\underline{k}\cdot\underline{p})^2}{\underline{k}^2} \right] \Big|_\infty$$

$$= 0 \qquad \ldots (G139)$$

Adding (G137), (G138) and (G139) gives

$$\Gamma^a_{0(ii)}(p)\Big|_\infty = 0 \qquad \ldots (G140)$$

$$\Gamma^a_{\ell(ii)}(p)\Big|_\infty :$$



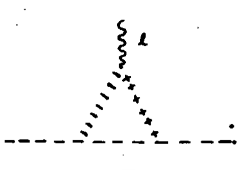
$$: \quad 2ig^2 i f^{abc} T^b T^c \int \frac{d^4k}{\underline{k}^4[(p-k)^2-m^2]} k_\ell (2p-k)_0^2 \Big|_\infty$$

$$= 2ig^2 i f^{abc} T^b T^c \int \frac{d^4k}{k^2\underline{k}^4} \left[1 + \frac{2k\cdot p}{k^2} + \cdots \right] k_\ell \left(k_0^2 - 4k_0p_0 + 4p_0^2 \right) \Big|_\infty$$

$$= 2ig^2 i f^{abc} T^b T^c \int \frac{d^4k}{k^2\underline{k}^4} \left[-2k_\ell(\underline{k}\cdot\underline{p}) \frac{k_0^2}{k^2} \right]$$

$$= -\tfrac{4}{3} p_\ell\, i g^2 i f^{abc} T^b T^c (2\pi)^4 I_{12}$$

$$= (2p_\ell)\left(-\tfrac{2}{3}\right) i f^{abc} T^b T^c L \qquad \ldots (G141)$$



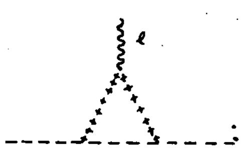
$$: \quad 4ig^2 i f^{abc} T^b T^c \int \frac{d^4k}{k^2\underline{k}^2[(p-k)^2-m^2]} (2k_0p_0 - k_0^2)\left(p_\ell - k_\ell \frac{\underline{k}\cdot\underline{p}}{\underline{k}^2}\right) \Big|_\infty$$

$$= 4ig^2 i f^{abc} T^b T^c \int \frac{d^4k}{k^4\underline{k}^2} \left[1 + \frac{2k\cdot p}{k^2} + \cdots \right] \left[p_\ell(2k_0p_0 - k_0^2) - k_\ell \frac{\underline{k}\cdot\underline{p}}{\underline{k}^2}(2k_0p_0 - k_0^2) \right] \Big|_\infty$$

$$= 4ig^2 i f^{abc} T^b T^c \int \frac{d^4k}{k^4\underline{k}^2} \left[-p_\ell k_0^2 + k_\ell(\underline{k}\cdot\underline{p}) \frac{k_0^2}{\underline{k}^2} \right]$$

$$= -\tfrac{8}{3} p\, i g^2 i f^{abc} T^b T^c (2\pi)^4 I_{12}$$

$$= (2p_\ell)\left(-\tfrac{4}{3}\right) i f^{abc} T^b T^c L \qquad \ldots (G142)$$



$$: \quad -8ig^2 i f^{abc} T^b T^c \int \frac{d^4k}{\underline{k}^4[(p-k)^2-m^2]} k_\ell \left[\underline{p}^2 - \frac{(\underline{k}\cdot\underline{p})^2}{\underline{k}^2} \right] \Big|_\infty$$

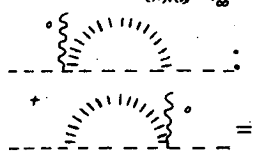
$$= 0 \qquad \ldots (G143)$$

Adding (G141), (G142) and (G143) gives

$$\Gamma^{a}_{\ell(ii)}(p)\Big|_{\infty} = (2p_\ell)(-2)\, i f^{abc}T^bT^c L \qquad \text{...(G144)}$$

$$\Gamma^{a}_{\substack{\lambda(iv)+(v)\\(p=p')}}(p)\Big|_{\infty} = -2ig^2\left(d^{abc}T^bT^c + \tfrac{1}{n}T^a\right)\int \frac{d^4k}{[(p+k)^2-m^2]}(2p+k)^\mu \Delta_{\lambda\mu}(k)\Big|_{\infty} \qquad \text{...(G145)}$$

$\Gamma^{a}_{0(iv)+(v)}(p)\Big|_{\infty}$:



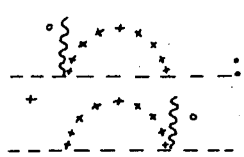
L

$$: \; -2ig^2\left(d^{abc}T^bT^c + \tfrac{1}{n}T^a\right)\int \frac{d^4k}{k^2[(p+k)^2-m^2]}(2p+k)_0\Big|_{\infty}$$

$$= -2ig^2\left(d^{abc}T^bT^c + \tfrac{1}{n}T^a\right)\int \frac{d^4k}{k^2k^2}\left[1-\frac{2k\cdot p}{k^2}+\cdots\right](2p+k)_0\Big|_{\infty}$$

$$= -4p_0\, ig^2\left(d^{abc}T^bT^c + \tfrac{1}{n}T^a\right)(2\pi)^{-4}[I_{11} - I_{12}]$$

$$= (2p_0)(-2)\left(d^{abc}T^bT^c + \tfrac{1}{n}T^a\right)L \qquad \text{...(G146)}$$



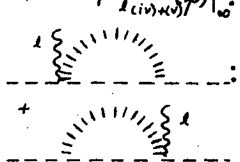
T

$$: \; 0 \qquad \text{...(G147)}$$

Adding (G146) and (G147) gives

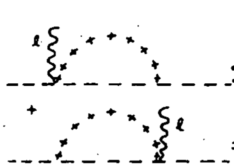
$$\Gamma^{a}_{0(iv)+(v)}(p)\Big|_{\infty} = (2p_0)(-2)\left(d^{abc}T^bT^c + \tfrac{1}{n}T^a\right)L \qquad \text{...(G148)}$$

$\Gamma^{a}_{\ell(iv)+(v)}(p)\Big|_{\infty}$:



L

$$: \; 0 \qquad \text{...(G149)}$$



T

$$: \; 2ig^2\left(d^{abc}T^bT^c + \tfrac{1}{n}T^a\right)\int \frac{d^4k}{k^2[(p+k)^2-m^2]}(2p+k)_m\left(\delta_{m\ell} - \frac{k_mk_\ell}{k^2}\right)\Big|_{\infty}$$

$$= 4ig^2\left(d^{abc}T^bT^c + \tfrac{1}{n}T^a\right)\int \frac{d^4k}{k^4}\left[1-\frac{2k\cdot p}{k^2}+\cdots\right]\left[p_\ell - k_\ell\frac{k\cdot p}{k^2}\right]\Big|_{\infty}$$

$$= \tfrac{8}{3}p_\ell\, ig^2\left(d^{abc}T^bT^c + \tfrac{1}{n}T^a\right)(2\pi)^{-4}I_1$$

$$= (2p_\ell)\left(-\tfrac{4}{3}\right)\left(d^{abc}T^bT^c + \tfrac{1}{n}T^a\right)L \qquad \text{...(G150)}$$

Adding (G149) and (G150) gives

$$\Gamma^a_{\ell(i)(ii)}(p)\Big|_\infty = (2p_\ell)\left(\frac{-4}{3}\right)\left(d^{abc}T^bT^c + \frac{1}{n}T^a\right)L \qquad \text{...(G151)}$$

Adding (G131), (G140) and (G148) gives

$$\Gamma^a_{\substack{0\\(total)}}(p)\Big|_\infty = (2p_0)\left[2T^bT^aT^b - 2\left(d^{abc}T^bT^c + \frac{1}{n}T^a\right)\right]L \qquad \text{...(G152a)}$$

$$= (2p_0)\left[\frac{-(n^2-1)}{n}\right]T^aL \qquad \text{...(G152b)}$$

Adding (G135), (G144) and (G151) gives

$$\Gamma^a_{\substack{\ell\\(total)}}(p)\Big|_\infty = (2p_\ell)\left[\frac{2}{3}T^bT^aT^b - 2if^{abc}T^bT^c - \frac{4}{3}\left(d^{abc}T^bT^c + \frac{1}{n}T^a\right)\right]L \qquad \text{...(G153a)}$$

$$= (2p_\ell)\left[\frac{n^2+3}{3n}\right]T^aL \qquad \text{...(G153b)}$$

where (B16), (B17) and (B18) have been used.

FERMION VERTEX CORRECTION:

Again $p=p'$, and the results are proportional to γ_0 and γ_ℓ respectively.

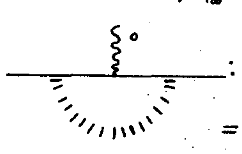
$$\Gamma^a_{\substack{\lambda(i)\\(p=p')}}(p)\Big|_\infty = i g^2 T^bT^aT^b \int \frac{d^4k}{k^2[(p+k)^2-m^2]} \times$$

$$\times \left\{ \frac{k^2}{\underline{k}^2}\gamma_0(\not p+\not k+m)\gamma_\lambda(\not p+\not k+m)\gamma_0 + \left(\delta_{ij}-\frac{k_ik_j}{\underline{k}^2}\right)\gamma_i(\not p+\not k+m)\gamma_\lambda(\not p+\not k+m)\gamma_j\right\}\Big|_\infty$$

$$= i g^2 T^bT^aT^b \int \frac{d^4k}{k^6}\left[\frac{k^2}{\underline{k}^2}\gamma_0\not k\gamma_\lambda\not k\gamma_0 + \left(\delta_{ij}-\frac{k_ik_j}{\underline{k}^2}\right)\gamma_i\not k\gamma_\lambda\not k\gamma_j\right]$$

...(G154)

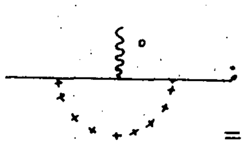
$\Gamma^a_{0(i)}(p)\Big|_\infty$: Using (E29), (E32) and (E33) we have:



$$: \; i g^2 T^bT^aT^b \int \frac{dk}{k^4\underline{k}^2}\left[\gamma_0 k_0 + 2k_0\underline{\gamma}\cdot\underline{k} + \gamma_0\underline{k}^2\right]$$

$$= \gamma_0\, i g^2 T^bT^aT^b(2\pi)^4\left[I_{12} + I_1\right]$$

$$= 0 \qquad \text{...(G155)}$$



$$: \; i g^2 T^bT^aT^b \int \frac{d^4k}{k^6}\left[2\gamma_0 k_0 + 2\gamma_0\underline{k}^2 - 4k_0\underline{\gamma}\cdot\underline{k}\right]$$

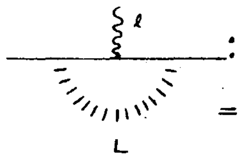
$$= 2\gamma_0\, i g^2 T^bT^aT^b(2\pi)^{-4}\left[I_3 + I_2\right]$$

$$= \gamma_0\, T^bT^aT^b\, L \qquad \text{...(G156)}$$

Adding (G155) and (G156) gives

$$\Gamma^a_{0(i)}(p)\Big|_\infty = \gamma_0\, T^bT^aT^b L \qquad \text{...(G157)}$$

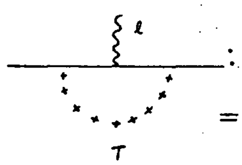
$\Gamma^a_{\ell(i)}(p)\big|_\infty$: Using (E30), (E35) and (E36) we have:



$$: \; i g^2 T^b T^a T^b \int \frac{d^4k}{k^4 \underline{k}^2} \left[\gamma_\ell k^2 + 2\gamma_0 k_0 k_\ell + 2 k_\ell \underline{\gamma}\cdot\underline{k} \right]$$

$$= \gamma_\ell \, i g^2 T^b T^a T^b (2\pi)^{-4} \left[I_{11} + \tfrac{2}{3} I_1 \right]$$

$$= \gamma_\ell \, \tfrac{4}{3} \, T^b T^a T^b L \qquad \ldots(\text{G158})$$



$$: \; i g^2 T^b T^a T^b \int \frac{d^4k}{k^6} \left[-4\gamma_0 k_0 k_\ell - 2 k_\ell \underline{\gamma}\cdot\underline{k} - 2 k_\ell \underline{\gamma}\cdot\underline{k} \, \frac{k_0^2}{\underline{k}^2} \right]$$

$$= -\tfrac{2}{3} \gamma_\ell \, i g^2 T^b T^a T^b (2\pi)^{-4} \left[I_2 + I_3 \right]$$

$$= \gamma_\ell \left(-\tfrac{1}{3}\right) T^b T^a T^b L \qquad \ldots(\text{G159})$$

Adding (G158) and (G159) gives

$$\Gamma^a_{\ell(i)}(p)\big|_\infty = \gamma_\ell \, T^b T^a T^b L \qquad \ldots(\text{G160})$$

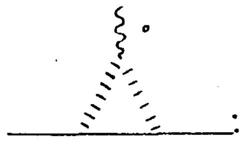
$$\Gamma^a_{\lambda(i)}(p)\big|_\infty \underset{(p=p')}{=} -i g^2 i f^{abc} T^b T^c \int \frac{d^4k}{(k^2-m^2)} \gamma_\mu (\not{k}+m) \gamma_\nu \, \Delta^{\mu\sigma}(p-k) \Delta^{\rho\nu}(p-k) \times$$
$$\times \left[-(2p-2k)_\lambda \eta_{\rho\sigma} + (p-k)_\rho \eta_{\sigma\lambda} + (p-k)_\sigma \eta_{\lambda\rho} \right]\Big|_\infty$$

$$= -i g^2 i f^{abc} T^b T^c \int \frac{d^4k}{k^2} \gamma_\mu \not{k} \gamma_\nu \Delta^{\mu\sigma}(k) \Delta^{\rho\nu}(k) \left[2k_\lambda \eta_{\rho\sigma} - k_\rho \eta_{\sigma\lambda} - k_\sigma \eta_{\lambda\rho} \right]$$

$$= -i g^2 i f^{abc} T^b T^c \int \frac{d^4k}{k^2} \times$$
$$\times \Big\{ \gamma_0 \not{k} \gamma_0 \frac{1}{\underline{k}^4} \left(2k_\lambda - 2k_0 \eta_{0\lambda} \right)\Big|_{LL}$$
$$+ \left(\gamma_0 \not{k} \gamma_n + \gamma_n \not{k} \gamma_0 \right) \frac{1}{k^2 \underline{k}^2} \left(\delta_{rn} - \frac{k_r k_n}{\underline{k}^2} \right) \left(-k_r \eta_{0\lambda} - k_0 \eta_{\lambda r} \right)\Big|_{LT+TL}$$
$$+ \gamma_m \not{k} \gamma_n \frac{1}{k^4} \left(\delta_{ms} - \frac{k_m k_s}{\underline{k}^2} \right) \left(\delta_{rn} - \frac{k_r k_n}{\underline{k}^2} \right) \left(-2k_\lambda \delta_{rs} - k_r \eta_{s\lambda} - k_s \eta_{\lambda r} \right)\Big|_{TT} \Big\}$$

$$\ldots(\text{G161})$$

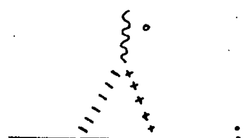
$\Gamma^a_{0(i)}(p)\big|_\infty$: Using (E23) and (E24) we have:



LL

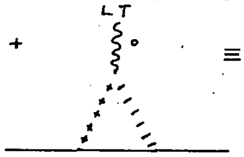
$$: \quad -ig^2 i f^{abc} T^b T^c \int \frac{d^4k}{k^2 \underline{k}^4} \gamma_0 \not{k} \gamma_0 (2k_0 - 2k_0)$$

$$\equiv \quad 0 \qquad \ldots(\text{G162})$$



LT

+



TL

$$: \quad -ig^2 i f^{abc} T^b T^c \int \frac{d^4k}{k^4 \underline{k}^2} (\gamma_0 \not{k} \gamma_n + \gamma_n \not{k} \gamma_0)(k_n - k_n)$$

$$\equiv \quad 0 \qquad \ldots(\text{G163})$$

TT

$$: \quad 2ig^2 i f^{abc} T^b T^c \int \frac{d^4k}{k^6} k_0 \gamma_m \not{k} \gamma_n \left(\delta_{mr} - \frac{k_m k_r}{\underline{k}^2}\right)\left(\delta_{nr} - \frac{k_n k_r}{\underline{k}^2}\right)$$

$$= 2ig^2 i f^{abc} T^b T^c \int \frac{d^4k}{k^6} \left[2\gamma_0 k_0^2 - 2k_0 \underline{\gamma}\cdot\underline{k}\right]$$

$$= 4\gamma_0 i g^2 i f^{abc} T^b T^c (2\pi)^{-4} I_3$$

$$= \gamma_0 (-1) i f^{abc} T^b T^c L \qquad \ldots(\text{G164})$$

Adding (G162), (G163) and (G164) gives

$$\Gamma^a_{0(i)}(p)\big|_\infty = \gamma_0 (-1) i f^{abc} T^b T^c L \qquad \ldots(\text{G165})$$

$\Gamma^a_{\ell(i)}(p)\big|_\infty$: Using (E20), (E21), (E23) and (E24) we have:

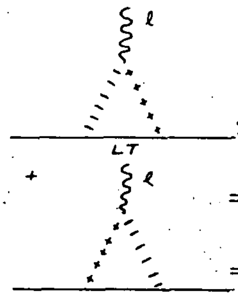
LL

$$: \quad -ig^2 i f^{abc} T^b T^c \int \frac{d^4k}{k^2 \underline{k}^4} \, 2k_\ell \gamma_0 \not{k} \gamma_0$$

$$= -2ig^2 i f^{abc} T^b T^c \int \frac{d^4k}{k^2 \underline{k}^4} k_\ell \left[\gamma_0 k_0 + \underline{\gamma}\cdot\underline{k}\right]$$

$$= -\frac{2}{3} \gamma_\ell i g^2 i f^{abc} T^b T^c (2\pi)^{-4} I_{11}$$

$$= \gamma_\ell \left(-\frac{4}{3}\right) i f^{abc} T^b T^c L \qquad \ldots(\text{G166})$$



LT + TL

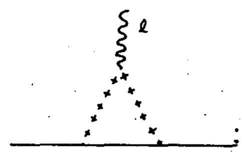
$$: \quad -ig^2 i f^{abc}T^bT^c \int \frac{d^4k}{k^4\underline{k}^2}\, k_0 \left(\gamma_0 \not{k} \gamma_n + \gamma_n \not{k} \gamma_0\right)\left(\delta_{\ell n} - \frac{k_\ell k_n}{\underline{k}^2}\right)$$

$$= -2ig^2 i f^{abc}T^bT^c \int \frac{d^4k}{k^4\underline{k}^2}\, k_0 \left(k_0\gamma_n + \gamma_0 k_n\right)\left(\delta_{\ell n} - \frac{k_\ell k_n}{\underline{k}^2}\right)$$

$$= -2ig^2 i f^{abc}T^bT^c \int \frac{d^4k}{k^4\underline{k}^2}\, k_0^2 \left(\gamma_\ell - k_\ell \frac{\underline{\gamma}\cdot\underline{k}}{\underline{k}^2}\right)$$

$$= -\tfrac{4}{3}\, i g^2 i f^{abc}T^bT^c (2\pi)^{-4} I_{12}$$

$$= \gamma_\ell \left(-\tfrac{4}{3}\right) i f^{abc}T^bT^c L \qquad \text{...(G167)}$$



TT

$$: \quad -ig^2 i f^{abc}T^bT^c \int \frac{d^4k}{k^6}\, \gamma_m \not{k} \gamma_n \left(-2k_\ell\delta_{rs} + k_r\delta_{s\ell} + k_s\delta_{\ell r}\right) \times$$

$$\times \left(\delta_{ms} - \frac{k_m k_s}{\underline{k}^2}\right)\left(\delta_{rn} - \frac{k_r k_n}{\underline{k}^2}\right)$$

$$= 2ig^2 i f^{abc}T^bT^c \int \frac{d^4k}{k^6} \left(-\gamma_0 k_0 \gamma_m\gamma_n + 2k_m\gamma_n + \underline{\gamma}\cdot\underline{k}\,\gamma_m\gamma_n\right)\left(k_\ell\delta_{mn} - \frac{k_\ell k_m k_n}{\underline{k}^2}\right)$$

$$= 4ig^2 i f^{abc}T^bT^c \int \frac{d^4k}{k^6} \left(\gamma_0 k_0 k_\ell - k_\ell\, \underline{\gamma}\cdot\underline{k}\right)$$

$$= -\tfrac{4}{3}\, \gamma_\ell\, i g^2 i f^{abc}T^bT^c (2\pi)^{-4} I_2$$

$$= \gamma_\ell (-1)\, i f^{abc}T^bT^c L \qquad \text{...(G168)}$$

Adding (G166), (G167) and (G168) gives

$$\Gamma^a_{\ell(i)}(p)\Big|_\infty = \gamma_\ell \left(\tfrac{-11}{3}\right) i f^{abc}T^bT^c L \qquad \text{...(G169)}$$

Adding (G157) and (G164) gives

$$\underset{(total)}{\Gamma^a_0(p)}\Big|_\infty = \gamma_0 \left[T^bT^aT^b - i f^{abc}T^bT^c\right] L \qquad \text{...(G170a)}$$

$$= \gamma_0 \left[\frac{n^2-1}{2n}\right] T^a L \qquad \text{...(G170b)}$$

Adding (G159) and (G168) gives

$$\underset{(total)}{\Gamma^a_\ell(p)}\Big|_\infty = \gamma_\ell \left[T^bT^aT^b - \tfrac{11}{3}\, i f^{abc}T^bT^c\right] L \qquad \text{...(G171a)}$$

$$= \gamma_\ell \left[\frac{11n^2-3}{6n}\right] T^a L \qquad \text{...(G171b)}$$

where (B16) and (B17) have been used.

GHOST SELF-ENERGY:

There is no coupling between the ghost field and $\Delta_{00}(k)$.

Thus

$$\Pi^{aa'}(p)\Big|_{\infty} = i\, g^2 C_A\, \delta^{aa'} \int \frac{d^4k}{(\mathbf{p}+\mathbf{k})^2} \left[(p+k)_\mu - (p+k)_0\, \eta_{\mu 0} \right] \left(p_\nu - p_0\, \eta_{\nu 0} \right) \Delta^{\mu\nu}(k)\Big|_{\infty} \quad \ldots \text{(G172)}$$

$$= i\, g^2 C_A\, \delta^{aa'} \int \frac{d^4k}{k^2(\mathbf{p}+\mathbf{k})^2} (p+k)_m\, p_n \left(\delta_{mn} - \frac{k_m k_n}{\mathbf{k}^2} \right)\Big|_{\infty}$$

$$= i\, g^2 C_A\, \delta^{aa'} \int \frac{d^4k}{k^2 \mathbf{k}^2} \left[1 - \frac{2\mathbf{k}\cdot\mathbf{p}}{\mathbf{k}^2} + \cdots \right] \left[\mathbf{p}^2 - \frac{(\mathbf{p}\cdot\mathbf{k})^2}{\mathbf{k}^2} \right]\Big|_{\infty}$$

T

$$= \frac{2}{3}\, \mathbf{p}^2\, i\, g^2\, C_A\, \delta^{aa'} (2\pi)^{-4} I_{11}$$

$$= \mathbf{p}^2 \left(\tfrac{4}{3} \right) C_A\, \delta^{aa'} L \quad \ldots \text{(G173)}$$

GHOST VERTEX CORRECTION:

The number of diagrams is reduced due to the decoupling of $\Delta_{00}(k)$ from the ghost field.

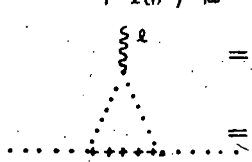
$$\Gamma^{abc}_{\lambda(i)}(p)\Big|_{\infty \atop (p=p')} = -\frac{1}{2} g^2 C_A f^{abc} \int \frac{d^4k}{(\mathbf{p}+\mathbf{k})^4} \left[(p+k)_\mu - (p+k)_0\, \eta_{\mu 0} \right] \left[(p+k)_\lambda - (p+k)_0\, \eta_{\lambda 0} \right] \left(p_\nu - p_0\, \eta_{\nu 0} \right) \Delta^{\mu\nu}(k)\Big|_{\infty}$$

$$\ldots \text{(G174)}$$

So $\Gamma^{abc}_{0(i)}(p)\Big|_{\infty} \equiv 0 \quad \ldots$(G175)

and

$$\Gamma^{abc}_{\ell(i)}(p)\Big|_{\infty} = -\frac{1}{2} g^2 C_A f^{abc} \int \frac{d^4k}{k^2(\mathbf{p}+\mathbf{k})^4} (p+k)_m (p+k)_\ell\, p_n \left(\delta_{mn} - \frac{k_m k_n}{\mathbf{k}^2} \right)\Big|_{\infty}$$



$$= -\frac{1}{2} g^2 C_A f^{abc} \int \frac{d^4k}{k^2(\mathbf{p}+\mathbf{k})^4} (p+k)_\ell \left[\mathbf{p}^2 - \frac{(\mathbf{p}\cdot\mathbf{k})^2}{\mathbf{k}^2} \right]\Big|_{\infty}$$

$$= 0 \quad \ldots \text{(G176)}$$

T

$$\Gamma^{abc}_{\lambda(ii)}(p)\Big|_{\infty \atop (p=p')} = \frac{1}{2} g^2 C_A f^{abc} \int \frac{d^4k}{\mathbf{k}^2} \left(k_\mu - k_0\, \eta_{\mu 0} \right) \left(p_\nu - p_0\, \eta_{\nu 0} \right) \Delta^{\mu\mu'}(p-k)\, \Delta^{\nu\nu'}(p-k) \times$$

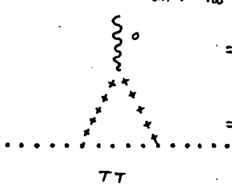
$$\times \left[(p-k)_{\mu'}\, \eta_{\lambda\nu'} + 2(k-p)_\lambda\, \eta_{\nu'\mu'} + (p-k)_{\nu'}\, \eta_{\mu'\lambda} \right]\Big|_{\infty}$$

$$= \frac{1}{2} g^2 C_A f^{abc} \int \frac{d^4k}{(\mathbf{p}-\mathbf{k})^2} \left[(p-k)_\mu - (p-k)_0 \eta_{\mu 0} \right] (p_\nu - p_0 \eta_{\nu 0}) \Delta^{\mu\mu'}(k) \Delta^{\nu\nu'}(k) \times$$

$$\times \left[k_{\mu'} \eta_{\lambda\nu'} - 2k_\lambda \eta_{\nu'\mu'} + k_{\nu'} \eta_{\mu'\lambda} \right] \Big|_\infty \qquad \ldots (G177)$$

So

$$\Gamma^{abc}_{o(ii)}(p)\Big|_\infty = -g^2 C_A f^{abc} \int \frac{d^4k}{k^4(\mathbf{p}-\mathbf{k})^2} (p-k)_m p_n k_0 \left(\delta_{mn} - \frac{k_m k_n}{\mathbf{k}^2} \right) \Big|_\infty$$



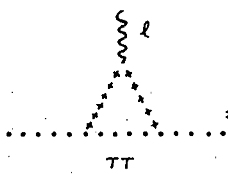
$$= -g^2 C_A f^{abc} \int \frac{d^4k}{k^4(\mathbf{p}-\mathbf{k})^2} k_0 \left[\mathbf{p}^2 - \frac{(\mathbf{p}\cdot\mathbf{k})^2}{\mathbf{k}^2} \right] \Big|_\infty$$

$$= 0 \qquad \ldots (G178)$$

and

$$\Gamma^{abc}_{\ell(ii)}(p)\Big|_\infty = -\frac{1}{2} g^2 C_A f^{abc} \int \frac{d^4k}{k^4(\mathbf{p}-\mathbf{k})^2} (p-k)_m p_n \left(\delta_{mm'} - \frac{k_m k_{m'}}{\mathbf{k}^2} \right) \left(\delta_{nn'} - \frac{k_n k_{n'}}{\mathbf{k}^2} \right) \times$$

$$\times \left[k_{m'} \delta_{\ell n'} - 2k_\ell \delta_{m'n'} + k_{n'} \delta_{m'\ell} \right] \Big|_\infty$$



$$= -\frac{1}{2} g^2 C_A f^{abc} \int \frac{d^4k}{k^4(\mathbf{p}-\mathbf{k})^2} (p-k)_m p_n \left[2k_\ell \frac{k_m k_n}{\mathbf{k}^2} - 2k_\ell \delta_{mn} \right] \Big|_\infty$$

$$= -\frac{1}{2} g^2 C_A f^{abc} \int \frac{d^4k}{k^4(\mathbf{p}-\mathbf{k})^2} k_\ell \left[\frac{(\mathbf{k}\cdot\mathbf{p})^2}{\mathbf{k}^2} - \mathbf{p}^2 \right] \Big|_\infty$$

$$= 0 \qquad \ldots (G179)$$

Adding (G175), (G176), (G177), (G178) and (G179) gives

$$\Gamma^{abc}_{\lambda\,(total)}(p)\Big|_\infty = 0 \qquad \ldots (G180)$$

APPENDIX H. DETAILS OF CALCULATIONS IN THE GENERAL GAUGE

The relevant diagrams and combinatorical coefficients are unchanged from Appendix G.

SCALAR MESON SELF-ENERGY:

$$\Pi(p)\Big|_\infty = -ig^2 C_\phi \int \frac{d^4k}{k^2[(p+k)^2-m^2]} (2p+k)^\mu (2p+k)^\nu \times$$
$$\times \left[-\eta_{\mu\nu} + \frac{(b^2-a^2)k_0}{(b^2k_0^2-a^2\underline{k}^2)}(k_\mu \eta_{\nu 0} + k_\nu \eta_{\mu 0}) - \frac{(1-a^4)k^2+(b^2-a^2)^2k_0^2}{(b^2k_0^2-a^2\underline{k}^2)^2} k_\mu k_\nu \right]\Bigg|_\infty \qquad \text{...(H1)}$$

$$= ig^2 C_\phi \int \frac{d^4k}{k^2[(p+k)^2-m^2]} \Big\{ (2p+k)^2 - 2\frac{(b^2-a^2)}{(b^2k_0^2-a^2\underline{k}^2)} k_0 (2p+k)_0 k.(2p+k)$$
$$+ \frac{(1-a^4)k^2+(b^2-a^2)^2k_0^2}{(b^2k_0^2-a^2\underline{k}^2)^2}[k.(2p+k)]^2 \Big\}\Big|_\infty$$

$$= ig^2 C_\phi \int \frac{d^4k}{k^4}\left[1 - \frac{2k.p}{k^2} + \frac{4(k.p)^2}{k^4} + \dots\right] \times$$
$$\times \Big\{ k^2 + 4k.p + 4p^2$$
$$-2\frac{(b^2-a^2)}{(b^2k_0^2-a^2\underline{k}^2)}\left[4p_0^2k_0^2 - 4p_0k_0(\underline{k}.\underline{p}) + 2p_0k_0k^2 + 2p_0k_0^3 - 2k_0^2(\underline{k}.\underline{p}) + k_0^2k^2\right]$$
$$+ \frac{(1-a^4)k^2+(b^2-a^2)^2k_0^2}{(b^2k_0^2-a^2\underline{k}^2)^2}\left[k^4 + 4p_0k_0k^2 - 4k^2(\underline{k}.\underline{p}) + 4p_0^2k_0^2 - 8p_0k_0(\underline{k}.\underline{p}) + 4(\underline{k}.\underline{p})^2\right]\Big\}\Big|_\infty$$

$$= ig^2 C_\phi \int \frac{d^4k}{k^4}\Big\{ -(p^2-m^2) - 4\frac{(k.p)^2}{k^2} + 4p^2$$
$$+ 2\frac{(b^2-a^2)}{(b^2k_0^2-a^2\underline{k}^2)}\left[k_0^2(p^2-m^2)\right]$$
$$+ \frac{(1-a^4)k^2+(b^2-a^2)^2k_0^2}{(b^2k_0^2-a^2\underline{k}^2)^2}\left[-k^2(p^2-m^2)\right] \Big\}$$

$$= ig^2 C_\phi (2\pi)^{-4}\Big\{ (2p^2+m^2) I_1 + 2(p^2-m^2)(b^2-a^2) J_6$$
$$- (p^2-m^2)\left[(1-a^4) J_1 + (b^2-a^2)^2 J_4\right] \Big\}$$

$$= \Big\{ -(2p^2+m^2) + (p^2-m^2)\Big[2\frac{(a-b)^2}{(a^2-b^2)} + \frac{1-a^4}{a^2b} + \frac{(a-b)^2}{ab}\Big]\Big\} C_\phi L$$

$$= \Big\{ -(2p^2+m^2) + (p^2-m^2)\Big[\frac{4b^2}{(a^2-b^2)} - \frac{4ab}{(a^2-b^2)} + \frac{b}{a} + \frac{1}{ab}\Big]\Big\} C_\phi L \qquad \text{...(H2)}$$

Thus the special limits are:

$$\Pi_{(cov)}(p)\Big|_{\infty} = \left[-(2p^2+m^2) - (p^2-m^2)(1-c) \right] C_\phi L \qquad \ldots (H3c)$$

and

$$\Pi_{(n.c.)}(p)\Big|_{\infty} = \left[-(2p^2+m^2) - (p^2-m^2)\frac{4b^2}{(b^2-a^2)} \right] C_\phi L \qquad \ldots (H3n)$$

FERMION SELF-ENERGY:

$$\begin{aligned}
\Sigma(p)\Big|_{\infty} &= -ig^2 C_\psi \int \frac{d^4k}{k^2[(p+k)^2-m^2]} \gamma^\mu (\not p + \not k + m)\gamma^\nu \times \\
&\quad \times \left[-\eta_{\mu\nu} + \frac{(b^2-a^2)k_0}{(b^2k_0^2-a^2\underline{k}^2)}(k_\mu\eta_{\nu 0}+k_\nu\eta_{\mu 0}) - \frac{(1-a^4)k^2+(b^2-a^2)^2k_0^2}{(b^2k_0^2-a^2\underline{k}^2)^2} k_\mu k_\nu \right]\Big|_{\infty} \\
&= i g^2 C_\psi \int \frac{d^4k}{k^2} \Big\{ \frac{\gamma^\mu(\not p+\not k+m)\gamma_\mu}{[(p+k)^2-m^2]} \\
&\qquad - \frac{(b^2-a^2)k_0}{(b^2k_0^2-a^2\underline{k}^2)}\left[\not k \frac{1}{\not p+\not k-m}\gamma_0 + \gamma_0 \frac{1}{\not p+\not k-m}\not k \right] \\
&\qquad + \frac{(1-a^4)k^2+(b^2-a^2)^2k_0^2}{(b^2k_0^2-a^2\underline{k}^2)^2} \not k \frac{1}{\not p+\not k-m}\not k \Big\}\Big|_{\infty} \qquad \ldots (H4) \\
&= i g^2 C_\psi \int \frac{d^4k}{k^2} \Big\{ \frac{1}{k^2}\left[1-\frac{2k.p}{k^2}+\cdots\right]\left[-2\not p-2\not k+4m\right] \\
&\qquad - \frac{(b^2-a^2)k_0}{(b^2k_0^2-a^2\underline{k}^2)}\left[\left(1-(\not p-m)\frac{\not k}{k^2}+\cdots\right)\gamma_0+\gamma_0\left(1-\frac{\not k}{k^2}(\not p-m)+\cdots\right)\right] \\
&\qquad + \frac{(1-a^4)k^2+(b^2-a^2)^2k_0^2}{(b^2k_0^2-a^2\underline{k}^2)^2}\not k\left[\frac{1}{\not k}-\frac{1}{\not k}(\not p-m)\frac{1}{\not k}+\cdots\right]\not k \Big\}\Big|_{\infty} \\
&= i g^2 C_\psi \int \frac{d^4k}{k^2} \Big\{ \frac{1}{k^2}\left[-2\not p+4\frac{\not k(k.p)}{k^2}+4m\right] \\
&\qquad + 2\frac{(b^2-a^2)}{k^2(b^2k_0^2-a^2\underline{k}^2)} k_0^2(\not p-m) \\
&\qquad - \frac{(1-a^4)k^2+(b^2-a^2)^2k_0^2}{(b^2k_0^2-a^2\underline{k}^2)^2}(\not p-m) \Big\}\Big|_{\infty} \\
&= i g^2 C_\psi (2\pi)^{-4}\Big\{ (-\not p+4m)I_1 + 2(\not p-m)(b^2-a^2)J_6 \\
&\qquad -(\not p-m)\left[(1-a^4)J_1 + (b^2-a^2)^2 J_4\right]\Big\} \\
&= \Big\{ \not p - 4m + (\not p-m)\left[\frac{2(a-b)^2}{(a^2-b^2)} + \frac{1-a^4}{a^3b} + \frac{(a-b)^2}{ab}\right]\Big\} C_\psi L \\
&= \Big\{ \not p - 4m + (\not p-m)\left[\frac{4b^2}{(a^2-b^2)} - \frac{4ab}{(a^2-b^2)} + \frac{b}{a} + \frac{1}{a^3b}\right]\Big\} C_\psi L
\end{aligned} \qquad \ldots (H5)$$

where (E26), (F7) and (F8) have been used.

Thus the special limits are:

$$\Sigma_{(cov)}(p)\Big|_{\infty} = [\not p - 4m - (\not p - m)(1-c)]\, C_\gamma L \qquad \text{...(H6c)}$$

and

$$\Sigma_{(n.c)}(p)\Big|_{\infty} = [\not p - 4m - (\not p - m)\frac{4b^2}{(b^2-a^2)}]\, C_\gamma L \qquad \text{...(H6n)}$$

GHOST SELF-ENERGY:

$$\Pi^{aa'}(p)\Big|_{\infty} = -i g^2 C_A \delta^{aa'} \int \frac{d^4k \; a^2}{[b^2(p+k)_0^2 - a^2(\underline{p}+\underline{k})^2]} \left[(p+k)_\mu + \frac{(b^2-a^2)}{a^2}(p+k)_0 \eta_{\mu 0}\right] \times$$

$$\times \left[p_\nu + \frac{(b^2-a^2)}{a^2} p_0 \eta_{\nu 0}\right] \Delta^{\mu\nu}(k)\Big|_{\infty} \qquad \text{...(H7)}$$

$$= -i g^2 C_A \delta^{aa'} \int \frac{d^4k \; a^2}{(b^2k_0^2 - a^2\underline{k}^2)} \left[1 - \frac{2b^2 k_0 p_0}{(b^2k_0^2 - a^2\underline{k}^2)} + \frac{2a^2 \underline{k}\cdot\underline{p}}{(b^2k_0^2-a^2\underline{k}^2)} + \cdots\right] \times$$

$$\times \Big\{ \frac{b^4}{a^4} \frac{a^4\underline{k}^2 - k_0^2}{(b^2k_0^2-a^2\underline{k}^2)^2} (p_0^2 + p_0 k_0)$$

$$- \frac{b^2}{a^2} \frac{a^2b^2-1}{(b^2k_0^2-a^2\underline{k}^2)^2} \left(k_0^2 \underline{k}\cdot\underline{p} + \underline{k}^2 k_0 p_0 + 2k_0 p_0 \underline{k}\cdot\underline{p}\right)$$

$$+ \frac{1}{\underline{k}^2}\left[\underline{p}^2 - \frac{(\underline{k}\cdot\underline{p})^2}{\underline{k}^2}\right] + \frac{b^4k_0^2 - \underline{k}^2}{(b^2k_0^2-a^2\underline{k}^2)^2}\left[\underline{k}\cdot\underline{p} + \frac{(\underline{k}\cdot\underline{p})^2}{\underline{k}^2}\right]\Big\}\Big|_{\infty}$$

$$= -i g^2 C_A \delta^{aa'} \int \frac{d^4k \; a^2}{(b^2k_0^2 - a^2\underline{k}^2)} \times$$

$$\times \Big\{ \frac{b^4}{a^4} \frac{a^4\underline{k}^2 - k_0^2}{(b^2k_0^2-a^2\underline{k}^2)^2} \left[p_0^2 - 2p_0^2 \frac{b^2k_0^2}{(b^2k_0^2-a^2\underline{k}^2)}\right]$$

$$- \frac{b^2}{a^2} \frac{a^2b^2-1}{(b^2k_0^2-a^2\underline{k}^2)^2} \left[2\frac{a^2(\underline{k}\cdot\underline{p})^2k_0^2}{(b^2k_0^2-a^2\underline{k}^2)} - 2p_0^2 \frac{b^2k_0^2\underline{k}^2}{(b^2k_0^2-a^2\underline{k}^2)}\right]$$

$$+ \frac{1}{\underline{k}^2}\left[\underline{p}^2 - \frac{(\underline{k}\cdot\underline{p})^2}{\underline{k}^2}\right] + \frac{b^4k_0^2 - \underline{k}^2}{(b^2k_0^2-a^2\underline{k}^2)^2}\left[2\frac{a^2(\underline{k}\cdot\underline{p})^2}{(b^2k_0^2-a^2\underline{k}^2)} + \frac{(\underline{k}\cdot\underline{p})^2}{\underline{k}^2}\right]\Big\}$$

$$= -i g^2 C_A \delta^{aa'} (2\pi)^{-4}\Big[p_0^2\left(a^2b^4 J_{13} - \frac{b^4}{a^2}J_{14} - 2b^4 J_{19} + 2\frac{b^6}{a^2}J_{20}\right)$$

$$+ \underline{p}^2\left(\frac{2}{3}b^2a^2 J_{19} + \frac{2}{3}a^2 J_2 - \frac{2}{3}a^4 J_{18} + \frac{1}{3}a^2b^4 J_{14} - \frac{1}{3}a^2 J_{13}\right)\Big]$$

$$= \Big\{p_0^2\left[-\frac{3}{4}\frac{b^3}{a^3} + \frac{1}{4}\frac{b}{a^3}\right] + \underline{p}^2\left[-\frac{1}{4}\frac{1}{a^3b} + \frac{1}{12}\frac{b}{a} + \frac{4}{3}\frac{a(a-b)}{(a^2-b^2)}\right]\Big\} C_A \delta^{aa'} L$$

$$\text{...(H8)}$$

Thus the special limits are:

$$\Pi^{aa'}_{(cov)}(p)\Big|_{\infty} = \frac{1}{2}\left[p_0^2(c-3) - \underline{p}^2(c-1)\right] C_A \delta^{aa'} L = p^2 \frac{(c-3)}{4} C_A \delta^{aa'} L \qquad \text{...(H9c)}$$

and

$$\Pi^{aa'}_{(n.c.)}(p)\Big|_{\infty} = \underline{p}^2\left[-\frac{4}{3}\frac{a^2}{(b^2-a^2)}\right] C_A \delta^{aa'} L \qquad \text{...(H9n)}$$

GHOST VERTEX CORRECTION:

When working at zero momentum transfer, the results are proportional to $(\frac{b^2}{a^2})p_0$ and p_ℓ respectively, since the bare vertex contains the factor $[p_\lambda + \frac{b^2-a^2}{a^2} p_0 \eta_{\lambda 0}]$.

$$\Gamma^{abc}_{\lambda(i)}(p)\Big|_{\infty \atop (p=p')} = -\tfrac{1}{2} g^2 C_A f^{abc} \int \frac{d^{2\ell}k \; a^4}{[b^2(p+k)_0^2 - a^2(\underline{p}+\underline{k})^2]^2} [(p+k)_u + \frac{(b^2-a^2)}{a^2}(p+k)_0 \eta_{u0}] \times$$
$$\times [(p+k)_\lambda + \frac{(b^2-a^2)}{a^2}(p+k)_0 \eta_{\lambda 0}][p_0 + \frac{(b^2-a^2)}{a^2} p_0 \eta_{00}] \, \Delta^{u0}(k)\Big|_\infty \quad \ldots (H10)$$
$$= -\tfrac{1}{2} g^2 C_A f^{abc} \int \frac{d^{2\ell}k \; a^4}{(b^2k_0^2 - a^2\underline{k}^2)^2} [k_\lambda + \frac{(b^2-a^2)}{a^2} k_0 \eta_{\lambda 0}] \times$$
$$\times \Big\{ \frac{b^4}{a^4} \frac{a^4\underline{k}^2 - k_0^2}{(b^2k_0^2 - a^2\underline{k}^2)^2} k_0 p_0 - \frac{b^2}{a^2} \frac{a^2b^2 - 1}{(b^2k_0^2 - a^2\underline{k}^2)^2} (k_0^2 \underline{k}.\underline{p} + \underline{k}^2 k_0 p_0)$$
$$+ \frac{b^4k_0^2 - \underline{k}^2}{(b^2k_0^2 - a^2\underline{k}^2)^2} \underline{k}.\underline{p} \Big\}$$

So
$$\Gamma^{abc}_{0(i)}(p)\Big|_\infty = (\frac{b^2}{a^2} p_0)(-\tfrac{1}{2}) g^2 C_A f^{abc} \int d^{2\ell}k \Big[\frac{b^4k_0^2(a^4\underline{k}^2 - k_0^2)}{(b^2k_0^2 - a^2\underline{k}^2)^4} - \frac{b^2a^2(a^2b^2-1)k_0^2\underline{k}^2}{(b^2k_0^2 - a^2\underline{k}^2)^4} \Big]$$
$$= (\frac{b^2}{a^2} p_0)(-\tfrac{1}{2}) g^2 C_A f^{abc} (2\pi)^{-4} [b^2a^2 J_{19} - b^4 J_{20}]$$
$$= i(\frac{b^2}{a^2} p_0) \Big[\frac{1}{8} \quad \frac{1}{a^3 b}\Big] C_A f^{abc} L \quad \ldots (H11)$$

and

$$\Gamma^{abc}_{\ell(i)}(p)\Big|_\infty = -\tfrac{1}{2} g^2 C_A f^{abc} \int \frac{d^{2\ell}k}{(b^2k_0^2 - a^2\underline{k}^2)^2} k_\ell \Big[-\frac{b^2}{a^2} \frac{a^2b^2-1}{(b^2k_0^2 - a^2\underline{k}^2)^2} k_0^2 \underline{k}.\underline{p} + \frac{b^4k_0^2 - \underline{k}^2}{(b^2k_0^2 - a^2\underline{k}^2)^2} \underline{k}.\underline{p} \Big]$$
$$= \tfrac{1}{6} p_\ell (-g^2) C_A f^{abc} \int d^{2\ell}k \Big[\frac{-b^2a^2(a^2b^2-1)k_0^2\underline{k}^2}{(b^2k_0^2 - a^2\underline{k}^2)^4} + \frac{a^4\underline{k}^2(b^4k_0^2 - \underline{k}^2)}{(b^2k_0^2 - a^2\underline{k}^2)^4} \Big]$$
$$= \tfrac{1}{6} p_\ell (-g^2) C_A f^{abc} (2\pi)^{-4} [b^2a^2 J_{19} - a^4 J_{18}]$$
$$= i p_\ell \Big[\frac{1}{8} \quad \frac{1}{a^3 b}\Big] C_A f^{abc} L \quad \ldots (H12)$$

$$\Gamma^{abc}_{\lambda(ii)}(p)\Big|_{\infty \atop (p=p')} = -\tfrac{1}{2} g^2 C_A f^{abc} \int \frac{d^{2\ell}k \; a^2}{(b^2k_0^2 - a^2\underline{k}^2)} [k_u + \frac{(b^2-a^2)}{a^2} k_0 \eta_{u0}][p_0 + \frac{(b^2-a^2)}{a^2} p_0 \eta_{00}] \times$$
$$\times [(p-k)_{u'} \eta_{\lambda v'} + 2(k-p)_\lambda \eta_{v u'} + (p-k)_{v'} \eta_{u'\lambda}] \, \Delta^{uu'}(p-k) \Delta^{0v'}(p-k)\Big|_\infty$$
$$= \tfrac{1}{2} g^2 C_A f^{abc} \int \frac{d^{2\ell}k \; a^2}{(b^2k_0^2 - a^2\underline{k}^2)} [p_0 + \frac{(b^2-a^2)}{a^2} p_0 \eta_{00}][k_u + \frac{(b^2-a^2)}{a^2} k_0 \eta_{u0}] \times$$
$$\times [k_{u'} \eta_{\lambda v'} - 2k_\lambda \eta_{v u'} + k_{v} \eta_{u'\lambda}] \, \Delta^{uu'}(k) \, \Delta^{0v'}(k)$$

$$= \frac{1}{2} g^2 C_A f^{abc} \int \frac{d^4k \, a^2}{(b^2k_0^2 - a^2\underline{k}^2)} \left[p_\nu + \frac{(b^2-a^2)}{a^2} p_0 \eta_{\nu 0}\right]\left[k_\mu + \frac{(b^2-a^2)}{a^2} k_0 \eta_{\mu 0}\right] K^{\mu\nu}{}_{\lambda}(k)$$

...(H13)

where $K^{\mu\nu}{}_{\lambda}(k) = k_{\mu'} \Delta^{\mu\mu'}(k) \Delta^{\nu}{}_{\lambda}(k) - 2k_\lambda \Delta^{\mu}{}_{\rho}(k) \Delta^{\nu\rho}(k) + k_{\nu'} \Delta^{\mu}{}_{\lambda}(k) \Delta^{\nu\nu'}(k)$ ...(H14)

$$\Gamma^{abc}_{\lambda(ii)}(p)\Big|_{\infty \atop (p=p')} = \frac{1}{2} g^2 C_A f^{abc} \int \frac{d^4k \, a^2}{(b^2k_0^2 - a^2\underline{k}^2)} \times$$

$$\times \left[\frac{b^4}{a^4} p_0 k_0 \, K_{00\lambda}(k) - \frac{b^2}{a^2} (p_0 k_n + p_n k_0) K_{n0\lambda}(k) + p_n k_m K_{mn\lambda}(k) \right]$$

...(H15)

The structure of $K_{\mu\nu\lambda}(k)$ is:

$$K_{00\lambda}(k) = 2 \left[k_0 \Delta_{00}(k) - k_i \Delta_{0i}(k) \right] \Delta_{0\lambda}(k) - 2k_\lambda \left[\Delta_{00}(k) \Delta_{00}(k) - \Delta_{0i}(k)\Delta_{0i}(k) \right]$$

$$K_{n0\lambda}(k) = K_{0n\lambda}(k)$$

$$= \left[k_0 \Delta_{n0}(k) - k_i \Delta_{ni}(k) \right] \Delta_{0\lambda}(k) - 2k_\lambda \left[\Delta_{n0}(k) \Delta_{00}(k) - \Delta_{ni}(k)\Delta_{0i}(k) \right]$$

$$+ \left[k_0 \Delta_{00}(k) - k_i \Delta_{0i}(k) \right] \Delta_{n\lambda}(k)$$

$$K_{mn\lambda}(k) = \left[k_0 \Delta_{m0}(k) - k_i \Delta_{mi}(k) \right] \Delta_{n\lambda}(k) - 2k_\lambda \left[\Delta_{m0}(k) \Delta_{n0}(k) - \Delta_{mi}(k)\Delta_{ni}(k) \right]$$

$$+ \left[k_0 \Delta_{n0}(k) - k_i \Delta_{ni}(k) \right] \Delta_{m\lambda}(k)$$

...(H16)

So
$$\Gamma^{abc}_{0(ii)}(p)\Big|_{\infty} = \frac{1}{2} g^2 C_A f^{abc} \int \frac{d^4k \, a^2}{(b^2k_0^2 - a^2\underline{k}^2)} \left[\frac{b^4}{a^4} p_0 k_0 K_{000}(k) - \frac{b^2}{a^2} p_0 k_n K_{n00}(k) \right]$$

$$= \frac{1}{2} g^2 C_A f^{abc} \int \frac{d^4k \, a^2}{(b^2k_0^2 - a^2\underline{k}^2)^5} \times$$

$$\times \Big\{ \frac{b^4}{a^4} p_0 \left[-2 \underline{k}^2 k_0^2 (a^2b^2-1)(a^4\underline{k}^2 - k_0^2) + 2 k_0^4 \underline{k}^2 (a^2b^2-1)^2 \right]$$

$$- \frac{b^2}{a^2} p_0 \left[-\underline{k}^2 (b^4k_0^2 - \underline{k}^2)(a^4\underline{k}^2 - k_0^2) + 2 k_0^2\underline{k}^2 (a^2b^2-1)(b^4k_0^2 - \underline{k}^2) \right.$$

$$\left. - \underline{k}^4 k_0^2 (a^2b^2-1)^2 \right] \Big\}$$

$$= \frac{1}{2} p_0 \, g^2 C_A f^{abc} \int \frac{d^4k \, a^2}{(b^2k_0^2 - a^2\underline{k}^2)^5} \times$$

$$\times \left[2 b^4a^2 k_0^2 \underline{k}^4 - b^6 k_0^4 \underline{k}^2 - a^4 b^2 \underline{k}^6 \right]$$

$$= \tfrac{1}{2} p_0 \; g^2 C_A f^{abc} (2\pi)^{-4} [2 b^4 a^2 J_{22} - b^6 J_{23} - a^4 b^2 J_{21}]$$

$$= i(\tfrac{b^2}{a^2} p_0) \; [\tfrac{3}{8} \tfrac{1}{a^3 b}] \; C_A f^{abc} L \qquad \ldots (H17)$$

and

$$\Gamma^{abc}_{\ell (ii)}(p)\Big|_\infty = \tfrac{1}{2} g^2 C_A f^{abc} \int \frac{d^4k \; a^2}{(b^2k_0^2 - a^2\underline{k}^2)} \left[-\frac{b^2}{a^2} p_n k_0 K_{n0\ell}(k) + p_n k_m K_{m\ell}(k) \right]$$

$$= \tfrac{1}{2} g^2 C_A f^{abc} \int \frac{d^4k \; a^2}{(b^2k_0^2 - a^2\underline{k}^2)} \times$$

$$\times \Big\{ -\frac{b^2}{a^2} \Big[\Big(k_0^2 \frac{a^4\underline{k}^2 - k_0^2}{(b^2k_0^2 - a^2\underline{k}^2)^2} - k_0^2\underline{k}^2 \frac{a^2b^2-1}{(b^2k_0^2-a^2\underline{k}^2)^2} \Big) \times$$

$$\times \Big(\frac{1}{\underline{k}^2}\Big(p_\ell - \frac{\underline{k}.\underline{p}\, k_\ell}{\underline{k}^2}\Big) + \frac{b^4k_0^2 - \underline{k}^2}{\underline{k}^2(b^2k_0^2 - a^2\underline{k}^2)^2} \underline{k}.\underline{p}\, k_\ell \Big)$$

$$- 2 k_\ell \underline{k}.\underline{p}\; k_0^2 \frac{(a^4\underline{k}^2 - k_0^2)(a^2b^2-1)}{(b^2k_0^2 - a^2\underline{k}^2)^4} + k_\ell \underline{k}.\underline{p}\; k_0^2 \frac{(a^2b^2-1)(b^4k_0^2 - \underline{k}^2)}{(b^2k_0^2-a^2\underline{k}^2)^4}$$

$$+ k_\ell \underline{k}.\underline{p}\;\; k_0^4 \frac{(a^2b^2-1)}{(b^2k_0^2 - a^2\underline{k}^2)^4} \Big]$$

$$+ \Big[k_0^2\underline{k}^2 \frac{a^2b^2-1}{(b^2k_0^2-a^2\underline{k}^2)^2} - \underline{k}^2 \frac{b^4k_0^2 - \underline{k}^2}{(b^2k_0^2-a^2\underline{k}^2)^2} \Big] \Big[\frac{1}{\underline{k}^2}\Big(p_\ell - \frac{\underline{k}.\underline{p}\,k_\ell}{\underline{k}^2}\Big) + \frac{b^4k_0^2 - \underline{k}^2}{\underline{k}^2(b^2k_0^2-a^2\underline{k}^2)^2} \underline{k}.\underline{p}\, k_\ell \Big]$$

$$- 2 k_\ell \underline{k}.\underline{p}\; k_0^2\underline{k}^2 \frac{(a^2b^2-1)^2}{(b^2k_0^2-a^2\underline{k}^2)^2} + k_\ell \underline{k}.\underline{p}\; \frac{(b^4k_0^2 - \underline{k}^2)^2}{(b^2k_0^2-a^2\underline{k}^2)^4}$$

$$+ k_\ell \underline{k}.\underline{p}\; k_0^2 \frac{(a^2b^2-1)(b^4k_0^2-\underline{k}^2)}{(b^2k_0^2-a^2\underline{k}^2)^4} \Big\}$$

$$= \tfrac{1}{6} p_\ell \;\; g^2 C_A f^{abc} \int d^{2\ell}k \;\; \times$$

$$\times \Big\{ 2 [b^2k_0^4 - (b^2+a^2)k_0^2\underline{k}^2 + a^2\underline{k}^4] \frac{1}{\underline{k}^2(b^2k_0^2-a^2\underline{k}^2)^3}$$

$$+ a^4b^2 \frac{k_0^2\underline{k}^4}{(b^2k_0^2-a^2\underline{k}^2)^5} - 2a^2b^4 \frac{k_0^4\underline{k}^2}{(b^2k_0^2-a^2\underline{k}^2)^5} + b^6 \frac{k_0^6}{(b^2k_0^2-a^2\underline{k}^2)^5} \Big\}$$

$$= \tfrac{1}{6} p_\ell \;\; g^2 C_A f^{abc} (2\pi)^{-4} \Big[2 (b^2 J_{17} - (b^2+a^2) J_{16} + a^2 J_{15})$$

$$+ ab\, J_{22} - 2ab\, J_{23} + b\, J_{24} \Big]$$

$$= i p_\ell [\tfrac{3}{8} \; \tfrac{1}{a^3 b}] \; C_A f^{abc} L \qquad \ldots (H18)$$

[since $b^2 J_{17} - (b^2+a^2) J_{16} + a^2 J_{15} = \frac{4}{a^3 b}$]

Combining (H11), (H12), (H17) and (H18) gives

$$\Gamma^{abc}_{\lambda\,(total)}(p)\Big|_\infty = i \Big[p_\lambda + \frac{(b^2-a^2)}{a^2} p_0 \eta_{\lambda 0} \Big] \Big[\tfrac{1}{2} \; \tfrac{1}{a^3 b} \Big] C_A f^{abc} L \qquad \ldots (H19)$$

REFERENCES - APPENDICES

1. A.J. Macfarlane, A. Sudbery and P.H. Weisz : Commun. Math. Phys. II, 77 (1978).

2. A.J. Macfarlane, A. Sudbery and P.H. Weisz : Proc. Roy. Soc. Lond. A314, 217 (1970).

3. J.M. Jauch and F. Rohrlich : *'The Theory of Photons and Electrons,'* Springer-Verlag (N.Y.) (1976).

4. J.D. Bjorken and S.D. Drell : *'Relativistic Quantum Field Theory,'* McGraw-Hill (1964).